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THE MATHEMATICAL GAZETTE

EDITED FOR THE MATHEMATICAL ASSOCIATION BY

T. A. A. BROADBENT, M.A.

ROYAL NAVAL COLLEGE, GREENWICH, LONDON, S.E. 10.

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Intending members of the Mathematical Association are requested to communicate with one of the Secretaries, G. L. Parsons, Roxby House, Paines Lane, Pinner, Middlesex; Miss M. E. Bowman, Maria Grey College, Brondesbury, London, N.W. 6. The subscription is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette", and each member receives a copy of each new Report as it is issued.

Change of Address should be notified to Mr. M. A. Porter, King Edward's School, Birmingham. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to have notice of this.

Subscriptions should be paid to the Hon. Treasurer, Mathematical Association, Gordon House, 29 Gordon Square, London, W.C. 1.

LOCAL BRANCHES.

In addition to the Annual Meeting of the Association, there are frequent meetings of the local branches.

Any member of the Association may be enrolled as a member of a local branch. Any other person may become an *associate* of a branch, subject to the approval of the local committee, on payment of a small subscription levied to meet local expenses. In the case of a member of the Association who is a member of a local branch, and whose subscriptions to the Association and to the branch are not in arrear, the local subscription is reduced, in the case of one branch only, by the sum of 1s. 6d., which is paid to the branch out of the general funds of the Association. An associate may attend the General Meetings of the Association, but may not vote on matters of business, and is not entitled to receive the publications of the Association or to use the Library. A branch which includes twenty-five full members of the Association is represented on the Council.

The Council of the Association will be glad to cooperate in the formation of others, wherever a number of persons interested in the teaching of mathematics wish to meet for the purpose of discussion and for mutual help.

Anyone willing to undertake the formation of such a branch is invited to communicate with one of the Honorary Secretaries of the Association.

A list of the local branches with their Honorary Secretaries is given below.

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| LONDON : | A. J. G. MAY, 51 Wychwood Avenue, Luton, Beds. |
| NORTH WALES : | S. MOSES, Lahore, Conway Road, Llandudno. |
| YORKSHIRE : | R. L. BOLT, Woodhouse Grove School, Apperley Bridge, Bradford. |
| BRISTOL : | DR. O. R. BALDWIN, 40 Salisbury Road, Redlands, Bristol, G. |
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| MIDLAND : | Miss L. E. HARDCASTLE, Holly Lodge Grammar School, Smethwick.
C. T. L. CATON, The Grammar School, Alcester. |
| NORTH-EASTERN : | J. W. BROOKS, 5 Holmfild Avenue, Harton, South Shields, Co. Durham. |
| LIVERPOOL : | Miss J. S. BATTY, Mathematics Department, The University, Liverpool. |
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| NORTHERN IRELAND : | Miss N. M. SAVAGE, 38 Cromwell Road, Belfast. |
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H. J. MELDRUM, The Teachers' College, Sydney. |
| QUEENSLAND : | J. P. MCCARTHY, The University of Queensland, Brisbane. |
| VICTORIA : | H. B. SARJEANT, 1 Eric Avenue, Mordialloc, Victoria, Australia. |
| AUCKLAND, N.Z. : | E. H. DRIVER, Auckland Grammar School, Auckland, New Zealand. |
| BIRMINGHAM UNIV. JUNIOR BRANCH : | Miss J. SANDERS, The University, Birmingham. |
| KING'S COLLEGE (LONDON) JUNIOR BRANCH : | A. G. BACON, 92 Gladstone Avenue, Manor Park, London, E. 12. |

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VOL. XXXI.

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THE MATHEMATICAL ASSOCIATION.

A GENERAL MEETING of the Mathematical Association was held at the Polytechnic, Regent Street, on 9th and 10th April, 1947.

On 9th April, the business meeting was held at 10.30 a.m., the President, Mr. W. F. Bushell, in the chair. The Report of the Council for 1946 was adopted.* The election of Professor G. B. Jeffery, D.Sc., F.R.S., as President for 1947, was announced. The following were elected as Vice-Presidents: Mr. W. L. Ferrar, Professor G. H. Hardy, Professor W. V. D. Hodge, Mr. W. Hope-Jones, Professor E. H. Neville, Mr. A. W. Siddons, Mr. C. O. Tuckey, Professor G. N. Watson, Professor Sir Edmund Whittaker, Professor A. N. Whitehead. The Treasurer, the Secretaries, the Librarian, the Editor of the *Mathematical Gazette*, and the Auditor were re-elected. The following were elected to serve on the Council: Mr. F. W. Kellaway, Dr. F. G. Maunsell, Dr. E. A. Maxwell, Mr. J. B. Morgan, Mr. M. A. Porter, Mr. A. Robson, Mrs. E. Shuttleworth, Mr. F. J. Swan, Mrs. E. M. Williams. The following amendment to the Rules, proposed by Mr. A. W. Siddons, seconded by Mr. A. Hedley Pope, was carried: "That Rule 36 (i) shall be amended to read as follows: 'There shall be an Annual General Meeting on such day or days as the Council may appoint'."

At 11 a.m., the President delivered his Presidential Address, "A Century of School Mathematics".† At 2 p.m., a discussion on "The Place of Visual Aids in the Teaching of Mathematics" was opened by Mr. I. R. Vesselo and Mr. G. Patrick Meredith. At 5 p.m., Professor H. Davenport, F.R.S., read a paper on "The Geometry of Numbers". In the evening, a *Conversazione* was held and was very successful.

On 10th April, at 10 a.m., Mr. B. C. Brookes read a paper on "The Incorporation of Statistics into a School Course", followed at 11 a.m. by Dr. R. L. Goodstein on "Proof by Reductio ad Absurdum". At 2 p.m., a discussion on "A Unified Course in Mathematics in Secondary Schools" was opened by Mr. A. W. Riley, Mr. K. S. Snell and Miss E. Barnes. At 5 p.m., Mr. P. F. Burns spoke on "The Teaching of Astronomy in Schools".

A Publishers' Exhibition and an Exhibition of Visual Aids material, the latter arranged in conjunction with the discussion held on 9th April, were open during the two days.

* See pp. 66-68.

† See pp. 69-89.

REPORT OF THE COUNCIL FOR THE YEAR 1946.

Membership.

During the year 1946, 257 new members were admitted, of whom 15 were junior members. The membership of the Association at the date of this report is 2312, of whom 5 are honorary members, 126 are life members, 2072 ordinary members and 109 junior members. There are also 40 Library members. During the latter part of the year a leaflet calling attention to the advantages of the Association has been widely circulated, and the Council wishes to record its thanks to the excellent and devoted labours of the Branch Secretaries who undertook the major part of the labour of distributing it and also to many Education Authorities who cooperated in the project in one way and another. The Council is pleased to report that as a consequence of this effort a large number of new members are joining the Association and that many of these come from types of school not previously represented in any strength. It is thought that the full result of the work of the Branch Secretaries has not yet been achieved and the Council looks forward to a further increase of membership during 1947.

The Council records with regret the death of the following members: Mr. J. A. Fewings (1924), Mr. D. Gibb (1923), Mr. H. Goddard (1912), Mr. A. Romney Green (1936), Mr. S. de J. Lenfestey (1904), Rev. S. J. Lister (1920), Mr. W. N. Maw (1937), Mr. H. Orfeur (1911), Mr. W. E. Paterson (1903) (Hon. Librarian 1922-3), Miss M. Punnett (1910) (Hon. Secretary 1912-1937), Mr. J. S. W. Usher (1935), Mr. J. T. Williams (1918). An appreciation of Miss Punnett's work for the Association has already appeared in the *Mathematical Gazette*.

Finance.

There has been an increase this year both in receipts and expenditure. The former is due to the rise in the number of members, and this should be more marked next year. The latter has three causes. *Gazettes* have increased in size and in cost of printing. All the main reports have had to be reprinted and this will react heavily on receipts in next year's accounts. All the sub-committees of the Teaching Committee have been active and hence their expenses have risen. Clearly some of the money saved in the last six years will need to be spent on the printing of new reports, but it is important that the other rises in expenditure should be met by greater income, and hence all members can help by enrolling new members.

The Branches.

The Branches Committee has held two meetings, and has provided a valuable medium for exchange of information, and for conveying ideas from the branches to the parent association and vice versa. Through the agency of this committee, the branches have done valuable work in contacting teachers of mathematics in secondary modern schools; this is bearing fruit in an increased and more broadly based membership of the Association.

All branches, with one exception, have now resumed activity on an almost pre-war scale, and there is a general reawakening of interest which augurs well for the future of the Association.

Most branches have discussed work in the secondary modern schools, and the Liverpool Branch has produced a common syllabus for the 11-13 age group, on which experimental work is being undertaken by a number of schools in the area.

The London Branch records a number of meetings with attendances of over 100, and a very valuable discussion on the Alternative School Certificate

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syllabus. The Plymouth Branch, with a discussion on Mathematics in the primary school and lectures on Nomograms and Relativity, is doing valuable pioneer work in the South-West. The Yorkshire Branch, which is conducting a vigorous recruiting campaign, has reluctantly said farewell to its President, Professor W. P. Milne, who was the guest of honour at the annual dinner.

Southampton has dealt with a variety of technical subjects, and had a well-attended display of mathematical films. The Midland Branch has made useful contacts with Training Colleges in the area, and has had meetings on John Wallis, Mathematical Geography, the use of a Mathematical Laboratory, etc. Manchester records an inspection of a differential analyser; the North-Eastern Branch a discussion on Technical Mathematics, while other branches report a number of valuable meetings.

The arrival of a food parcel, sent by the Victoria Branch to each of the English branches, was very much appreciated. The kindly thought behind the gift is another instance of the community of interest between Australian teachers of mathematics and their English colleagues.

All the overseas branches continue to do active work and there has been a considerable influx of new members to the Association from Australia. Reports of the proceedings at these branches have appeared in the inset to the *Mathematical Gazette*. An important event of the year has been the affiliation of the Auckland (New Zealand) Branch. The affiliation of this branch is largely due to its Secretary, Mr. E. H. Driver. The Council warmly welcomes the new branch and expresses the hope that its activities may further strengthen the many ties between teachers of mathematics in the two countries.

The Mathematical Gazette.

The difficulties in production and distribution, referred to in the Report for 1945, have not decreased, and the numbers are still lagging behind the scheduled publication dates.

The British Council has thanked the Association for its cooperation in the supply of micro-films of the *Gazette* to China during the latter part of the war.

Each number of the *Gazette* is being provided with a frontispiece of mathematical interest, following Professor Chapman's suggestion in his Presidential Address. Members can help by suggesting topics, and by indicating means for obtaining photographs.

Teaching Committee.

The new Teaching Committee, which holds office until 1950, held its first meeting immediately after the Annual General Meeting 1946. Eight sub-Committees were appointed and these have all embarked upon programmes of work within the framework of their respective terms of reference. The Trigonometry Committee has now completed the third draft of Part I of a new report on the teaching of this subject, and this should be available in print during the autumn. Part II of the report is also well-advanced. The Visual Aids Committee has arranged a comprehensive exhibition of all forms of visual aid for the Annual General Meeting 1947. The Preparatory Schools Committee has completed a revision of the Mathematical Syllabus for the Common Entrance Examination, while the Technical Schools Committee has in process a report on the training of teachers of mathematics in Technical Schools and another on the syllabus of work in these schools. The Sixth Form Committee is considering the relationship between the mathematics syllabus of the Advanced Course in the Schools and that of the first-year degree Course in the Universities. This Committee is also preparing scripts on Geometry and Calculus at the Sixth Form stage. In the Modern Schools and

Primary Schools Committees work is well advanced towards the preparation of new Reports on the Syllabus and the teaching methods for mathematics in these schools.

Five of the Mathematical Association Reports on the Teaching of Mathematics are still in print, and every endeavour will be made to keep them all available. The demand for these Reports is now sensational and reprints of them all have been completed during the year.

The Problem Bureau.

The Bureau continues to provide solutions for members, though not to quite the same extent as in previous years, and it is thought that the continued existence of this part of the work of the Association is possibly not known to all members. A number of members of the Association continue to give excellent service in the matter of providing solutions.

Some difficulty is encountered from time to time when members fail to return solutions sent to them. The Council appeals to all who use the Bureau to exercise care in this matter.

Officers and Council.

The Council wishes to place on record its appreciation of the services of Mr. W. F. Bushell, M.A., as President, and nominates Prof. G. B. Jeffery, Sc.D., F.R.S., as President for the ensuing year. Mr. A. W. Riley retires from the Council and the Council wishes to offer thanks to him for many years' service to the Association. A number of members other than the officers have assisted in the work of the Association. The Council returns thanks for all these services, particularly to Mr. J. C. Manisty, who has done much good work in the preparation of the Trigonometry Report and Mr. F. W. Kellaway who, in addition to giving much-valued assistance to the Editor of the *Mathematical Gazette*, has undertaken the Secretaryship of the Programme Committee.

Once more it is the Council's pleasant duty to record the indebtedness of the Association to Prof. Broadbent in connection with the *Mathematical Gazette*, to Prof. Neville for the work of the Library, to Mr. Gosset Tanner for conducting the Problem Bureau, as well as to the Treasurer and the Secretaries.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

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A CENTURY OF SCHOOL MATHEMATICS.

BY W. F. BUSHELL.

PRESIDENTIAL ADDRESS TO THE MATHEMATICAL ASSOCIATION,
APRIL, 1947.

I FEEL it a great honour to be permitted to address you to-day. There have been some schoolmaster Presidents in the history of the Association. R. B. Hayward was President from 1878 to 1888 while a master at Harrow; Canon J. M. Wilson, a pioneer of Science teaching at Rugby, and one of the earliest to work actively for geometrical reform, became our President long after his retirement, and then there have been three in recent times, with all of whom I have been closely associated, and to whom I should wish to pay tribute.

Mr. Siddons was President in 1935 and in his address he told us something of his debt to one he rightly called the real Father of this Association, Rawdon Levett. What Levett was to him, Mr. Siddons was to me, especially in my early days of teaching. I can never forget what I owe him. As a young man I used to put my problems before him. He was always sympathetic and helpful. In those early days of forty years ago when the whole method of teaching Mathematics was changing with great rapidity, both Mr. Godfrey and Mr. Siddons—a great partnership—stood supreme, as the pioneers of a new spirit. I have heard them described as the pall-bearers of Euclid; but they were more than that. They were pioneers of methods in all branches of Mathematics, which seem commonplace to-day, but which were then unknown or at all events rarely practised.

Then we came to the next schoolmaster President, my old friend, Mr. Hope-Jones. I have not forgotten, and I hope he has not, the three or four years we spent together at the same College, imbibing the same mathematical wisdom from a series of lecturers and tutors. He was then, as now, a man of enormous energy, and possessed of drive and powers of work. To-day he has often relieved the tedium of a Mathematical Council meeting by an apposite remark, or an appropriate quotation. It is no surprise to those of us who realise that it is the fulfilment of the promise he showed at that College of which he was so well known and illustrious a member.

And, lastly, the third schoolmaster President was Mr. Tuckey. He too has been a life-long friend. We both went to the same Public School in the same month of the same year. It is true that he remained there for over thirty years, while I deserted it after five. Our age difference then perhaps meant more than it does to-day, as at the time of our first acquaintance he was a young man and I a boy of thirteen. But I remember well the quickening influence he exercised over the Mathematics of that school, and what I personally owe to him. My knowledge of mathematics is limited. What I do know is largely due to his patient endeavours with an ordinary pupil. Indeed, but for those early efforts of his it is doubtful whether I should be occupying this chair to-day. I might add that I left school in 1903, just when Euclid was being dethroned, and well do I recollect how in my last term Mr. Tuckey produced a ruler with strange markings on it. It was, of course, a protractor. In those days, brought up on the purest Euclid, when protractors were banned and indeed unknown, I was dumbfounded at its appearance, and I remember quoting postulates and axioms and I know not what else, to demonstrate, I trust with courtesy, that its appearance to me, as a then conservative Euclidean, was both unwelcome and indeed disagreeable.

Thus there is a reason for my subject. I was educated under the old regime when Euclid was supreme and work, even in other elementary subjects, was

exceedingly formal and dull. Indeed, as I hope to show you later, methods of teaching in my schooldays were much the same as in the middle of last century. Reform was in the air but had not yet come. During the last half of the nineteenth century, I am convinced there was little change. It seems incredible that I finished a three years' course at the university in Mathematics without having entered a laboratory either at school or university, and yet we were taught subjects such as Hydrostatics, Optics, Electricity, Mechanics and the like. There were, of course, hundreds like me and I am sure we understood very little of what we were learning. We would at the university do strange problems in Optics, without having seen a lens. We would have to calculate the potential due to the most curious distributions of Electricity and yet be unable to handle an electric bell. It was an unreal and distorted education, and the university lecturers were said to delight in making problems of a ridiculous character. The most famous one, which was presumably quoted in order to amuse, was supposed to commence: "A Fly of Mass m is climbing up the trunk of an elephant whose mass may be neglected". Yet Euclid was sacred both at school and university. For instance, a Senior Wrangler of my time came out of his examination having forgotten the euclidean proof of a proposition set in the examination, and therefore seriously perturbed by the fact that he would get no credit for his non-euclidean proof. I myself took some steps later on at the university to remedy these deficiencies but I doubt if I or others ever fully recovered what we had lost.

Then my teaching as opposed to my learning days were entirely under what might be called the new dispensation, when Euclid had gone, and we were all seeking less formality in, and greater understanding of, the Mathematics we were passing on. Indeed events, when I commenced teaching in 1907, were moving like a flood. I claim, in consequence, by personal experience, knowledge of the old as well as the new methods of teaching and learning.

A century ago takes us to the days of Dr. Arnold. It was about his time, or rather earlier, that the better known schools first appointed voluntary Arithmetic masters. For instance, Mr. Marillier came to Harrow in 1819 for that purpose. It is perhaps rather strange that Mathematics was not made compulsory there till 1837 when we remember that Dr. George Butler, the great-grandfather of Mr. R. A. Butler of the Butler Act, was headmaster from 1805-1829 and had himself been Senior Wrangler. Apparently he contented himself with a very little Euclid reading in his sixth form.

But a century ago was an era of new textbooks and it was the coming of these that made the teaching of better Mathematics possible. If you look at the textbooks of the eighteenth and early nineteenth centuries you realise that they must always have repelled. "I attempted Mathematics", says the great Charles Darwin, "the work was repugnant to me, chiefly from my not being able to see any meaning in the earlier steps in Algebra." It is not surprising if you look at the textbooks of his schooldays.

We do not always recollect how enormously they have changed, and are still changing, for the better. Arithmetic was, of course, the oldest subject, and its development over the centuries might well be a matter for some future address. Indeed, in order to ensure continuity, I feel bound here to go back more than a century. It is popularly supposed that Robert Recorde made the first Arithmetic in English. But this is not so, as Cuthbert Tunstall completed an earlier one in 1522, a few days before his consecration as Bishop of London. Later on he was to be a principal figure in the persecution of William Tyndale, the first translator of the Bible into English.

Some time earlier, in 1511, he had been appointed Rector of Harrow, but this Arithmetic can hardly have benefited the boys of that famous school, as it was not to be founded for another half century. It was written too in Latin.

But I mention him because some of our modern problems in Arithmetic can trace an ancestry back to his date. He introduces for instance the problem of the cistern with three pipes emptying it in various times individually and it is required to find the time when they are all open together. Plumbing had to make a considerable advance before this develops into the two taps and waste pipes of nineteenth century textbooks. We find similar old friends in his partnership and other famous types of questions ; some of which, but not all, have survived to this day.

But Robert Recorde is far more famous. The only extant painting of him is to be found in the Mathematical School at Cambridge, and he is commemorated in the Parish Church of Tenby in Pembrokeshire, his birthplace, by a silhouette taken from that picture, and an inscription recording with pride the fact that he invented the two parallel lines as a sign of equality! This no doubt is true, but he did a great deal more. Though the *Whetstone of Wit* is his most famous book dealing with more advanced work, his Arithmetic textbook called the *Ground of Arts* was published in 1543, twenty years after Tunstall, and it was used for nearly two centuries, the last edition coming out in 1699. Perhaps it might be said he was the first to show how the rules of Arithmetic were used in the ordinary life of the people, and was the father of the books of Applied Arithmetic which have continued to the present century. This is not a history of Arithmetic, so I can do little more than mention the parents of our nineteenth century books.

But it is impossible to neglect Cocker, whose book, published in 1677, was only just ousted about a century ago, after a life of nearly two centuries. Indeed there was a 53rd edition in 1750.

Cocker was the parent of the nineteenth century books ; he himself largely copied from Recorde, and I myself believe Recorde and Cocker to have been the real forerunners of the Arithmetic of to-day. It was a book much criticised by the nineteenth century reformers and particularly by De Morgan who published his own *Arithmetic* in 1830, and who was one of the earliest to insist on what he called principles in his teaching. It is a very old criticism that boys are taught to obey rules rather than think in their Mathematics. But his book was not very successful as it was said to be too hard.

To Cocker was due the dot symbolism of the decimal point as we know it now. There were at least eight different ways of writing decimals before his time. It might be said that the teaching of decimals has never really been settled even to-day, and our modern books have always shown the decimal point hedged round with difficulties. We have all come across the boy who is not sufficiently conscious of the grave and serious error involved in getting his point wrong. Modern checking methods were introduced to correct this. It might be added that the history of the teaching of decimals since Recorde's time is in itself of considerable interest and repays study.

The book, however, that seems mainly to have been responsible for the disappearance of Cocker was by Colenso. I well remember an old Victorian schoolmaster asking me, as a boy, whether I worked at Cocker and Colenso. I imagine the question must have been a frivolous one. But Colenso was widely used in the year 1868 when the Schools Inquiry Commission published their report. Lists of books in use at the schools are given, and the Arithmetic and very often the Algebra by Colenso are commonly mentioned. Certainly it is in use far more than any other Arithmetic at the hundreds of schools on which the Commission reports, and I imagine at least half the pupils of the country must have been learning from it. Colenso was a master at Harrow from 1838-42, and I shall have more to say of him later. His book was a great improvement on that by Cocker, and although many other Arithmetics appeared during the century, which in their turn ousted Colenso, there was

really no notable change till 1900. I would again stress the fact that these great Arithmetics by Recorde, Cocker and Colenso all naturally take something from their predecessors, and "types of problems", to use an ancient phrase, have a long and interesting history.

In passing, I might add we find that in the 'sixties an even more universally used book is the Euclid by Todhunter. This was the best known of the famous Todhunter series, published in the middle of the nineteenth century or later, which started that long series of textbooks written by various eminent men on most elementary subjects.

Todhunter deserves a mention because of his influence on school work. He wrote, as I have said, a number of school textbooks as well as some learned ones. Indeed some lasted to my day. Senior Wrangler in 1848, and like Canon Wilson a Johnian, he was a great Cambridge coach, and produced many high Wranglers; he was a hard worker and rarely left his rooms in College. Once only did he see the May Bumping races. On subsequent occasions he declared that sitting in his study he could visualise exactly what was happening, and regarded it as quite unnecessary to attend in person. On his death in 1884, the *Cambridge Review* asked Professor Mayor, well known to many generations of Cambridge men, to write his biography. Mayor had a passion for completeness and accuracy but no sense of proportion, and in the first three instalments gave full details of the life and genealogy of three Dissenting ministers who had educated Todhunter. In despair the editor would allow no more to appear. Mayor and Todhunter were lifelong friends, yet it was said they had only once visited one another's rooms!

This twentieth century, the century of reform, has seen a great change in Arithmetic. General principles are much more freely taught. Weights and Measures have far less time assigned to them. Involved questions have vanished. Fractions commonly give way to decimals, and, in consequence, long H.C.F. by alternate division has disappeared. In short, Arithmetic is far more regarded as an introduction to further work. But only forty to fifty years ago books were still appearing with all sorts of types of questions dealing with ancient problems such as Tare and Tret, Alligation and others, which are treated in Robert Recorde's book and whose names are probably unknown to the teachers of to-day. They even invent new types such as cattle indulging in a contest with a field of uniformly growing grass.

But the greatest interest of these books of only half a century ago lies, firstly, in the absurd and pretended accuracy and, secondly, in the lack of reality in the questions. By lack of reality I have in mind the meaningless problems: "If ten boys catch six fish in seven days how long will it take a class of twenty-two boys to catch one fish." One can but wonder! Or "a fortress is supplied with 55,000 lbs. of bread and contains 800 men who eat 5 lbs. a week. How long will the provisions last?" My answer would be, it depends on the climate. There are hundreds like this. Or, as an example of pretended accuracy, we are told the dimensions and supposed accurate density of an iron tank, and asked to find its weight which is given as $231\frac{4376}{8293}$ lbs. There is a charm in that fraction. But these sort of answers are the rule, not the exception, and such went on to the end of last century, and indeed overflowed on to this.

Is it not possible that in approximations, logarithms and decimals instead of fractions we have the greatest change to-day in Arithmetic? And public understanding is on a higher level.

Forty years ago official statistics would actually tell us the value of the imports to this country in pounds to the nearest unit—the total figure amounting to several hundred millions. Further, at the same period, the Royal Commission on coal supplies gave the available resources of that part of

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the Cumberland coalfield lying between five and ten miles of the sea as 854,608,307 tons. Even when the Brooklands motor track opened in 1907 the length of the oval portion was officially announced as 2.71762547 miles to nine figures. Happily that could hardly occur to-day; but things are not perfect yet, as recently I saw that a form mistress in her terminal report quoted the average age of the form to three decimal places. I suppose the third place represents about eight hours!

I have been forced in saying these few words about Arithmetic to go back rather more than a century because naturally the development was continuous. The other great school subject called Euclid has a somewhat different history. For forty-five years he has gone, but my century also takes account of the sixty years before that when Euclid was dominant and methods of teaching changed little. It is a fact of considerable interest that England and her colonies retained Euclid long after other countries had made out and used a new system of Geometry. The present generation can perhaps hardly appreciate the story told me by a friend; he went to a German school in 1898 and told his Geometry master that he had done two books of Euclid. "Euclid" said the master, "What is that?" And then with a gleam of comprehension, "Oh, you mean the old Greek Geometrician". To our generation Euclid was known to all; a failure to recognise his name would have been incredible. Yet to a Geometry master in a German school even his name was scarcely known some fifty years ago.

Thus, though we were the last to desert him, it is also of interest that we were the first to adopt him. In the dark ages Euclid was unknown in western Europe. Greek versions did not exist there. The tradition was handed on by the Moorish universities in Spain. It is said that an English monk, Adelard of Bath, braved the perils of travel among the infidels and brought back to England an Arabic version of Euclid, which, translated into Latin, was the basis of geometrical learning here. It came from the press of Ratdolt at Venice in 1482 and contained several hundred illustrations. Adelard is a striking personality of the later middle ages, an embodiment of the new spirit which was animating the west. It is a striking fact that the spirit of the ancient learning comes to the west about this time and is received with enthusiasm; not only do we read of the coming of Euclid but Greece and all her glories are discovered. Yet, so weak is the spirit of man, that three to four centuries later the study of Greek had become for most of us little more than a mass of irregular verbs and nouns, and Euclid had become a text with which no one could tamper with safety! What is the reason for this insufficiency of man that the spirit so often gives way to the letter?

The earliest Greek edition of Euclid was printed by Johann Herwagen in Basle about 1533, and he was the first printer to insert Euclid's diagrams in the text. It is this edition which was depicted in the *Gazette* last October, and there is an introduction in Latin to Tunstall, the Bishop, and writer of the first Arithmetic book which I have already mentioned. The first translation into a modern language appears to be one of 1543 into Italian, but it is not till 1570 that the first edition* in English is published. Robert Recorde had previously given some enunciations, but not in Euclid's order, so to Henry Billingsley, citizen of London, must be given the honour of being the first English translator. Like many other great mathematicians he was at St. John's, Cambridge, and subsequently became Lord Mayor of London.

Many English editions have appeared since that date, but during the nineteenth century the main change was not indeed one of substance, but in the form of the textbook. It is fair to say that the better teachers endeavoured

* The frontispiece of this edition is reproduced in the *Gazette* for February, 1947. A copy (folio) was sold for £25 at Hodgson's auction sales in April, 1947.

by their presentation to make him more attractive, yet without altering his order, and scarcely interfering with his text.

It was really, however, the Victorian schoolmasters who placed Euclid on a pedestal he had scarcely occupied before. The classical example is that of Mr. Stephen Hawtreys who was appointed to Eton in 1836 by his cousin, Dr. Hawtreys, the headmaster. Like Marillier, at Harrow, there had been a writing and arithmetic master there in the person of Mr. Hexter who had got very old, and was a man of little influence. But Hawtreys was a far more striking figure. It is true that he came to give voluntary lessons only; as a contemporary says, "Mathematics at Eton were pretty much on a level with dancing and fencing as a mere extra study", and it was actually not till 1851 that Mathematics became compulsory, less than a century ago. He provided his own classroom; he provided and paid his own assistants, who were not regarded as proper masters. They could not wear a gown at chapel and had no powers of discipline outside school. Endless stories were told of them. One received a handsome present from his set because he allowed them to do as little as they pleased. Another reverend gentleman told his bad-mannered class that they forgot he was a "Man of God". This was long remembered. All this is only a century ago and we tend to forget how quickly things have moved. The Public Schools Commission of 1864, however, deprecated their position, and they became proper members of the staff in 1868; but certain disabilities were attached to them till the beginning of this century. It is worth adding that even in my schooldays certain masters for drawing and music were regarded as in some way inferior, and, as a boy, I unintentionally heard a prominent member of the staff regretting their admission to the masters' common room.

At Harrow this situation did not arise, but, up to a comparatively recent date, although Mathematical masters might become house masters, they could not act as tutors to their boys and had to import a Classical colleague.

The standard reached under Hawtreys was not high. We are told that Euclid and Algebra began in the fifth form, but a boy did not get there until he had a fair knowledge of Arithmetic, including unitary method, fractions and decimals. Afterwards a fair number read Trigonometry, very few advanced to Analytical Geometry, the highest point. Hawtreys confessed that he had never taught anyone the Differential Calculus.

In those days a solution by Calculus was considered rather mean if the problem could be solved by other complicated algebraical or geometrical methods, however cumbersome. Indeed this was also true of the Mathematical Tripos at Cambridge in the middle of last century, and later, as mathematicians of those days have told me; and further its presentation in the textbooks made it difficult and repellent to a boy. My generation, and indeed later ones, learnt it out of books which regarded the art of differentiating various functions as the main object of the science. And expressions of the greatest variety and complexity had to be differentiated before the pupil was allowed to learn the uses of the Calculus. In consequence, even in my time, no one understood much of what it meant, and, for a year or more, learners certainly knew little of its applications. To-day all the first emphasis is rightly placed on these uses, and at the beginning nothing save the simplest functions are differentiated. It always seems to me so strange that this, the most potent invention of Isaac Newton, and hence the intellectual inheritance of the British race, should not have made its way more easily into the schools.

But the above is a digression; Hawtreys himself survives because of his enthusiasm for Euclid. He had strong views on the value of the subject, and like other teachers of his date valued the deductive powers it was supposed to teach. He divided the books of the world into three classes: Class I, The

Bible; Class II, Euclid; Class III, all the rest. Further, he published a small book containing the first twelve propositions. The frontispiece consists of a curious figure with a number of interlocked rings, showing how the early propositions depend on one another. For instance, the eleventh proposition is joined to the third and the eighth, and so on. This was designed for him by his artist niece. The book is favourably reviewed in *The Times* of 1874, and many other journals, some of which are lyrical in their praises. One reviewer writes: "I cannot call to mind the case of a single pupil who found the study of Euclid repulsive. They may have found it hard, but all of them have felt that there was something real and great in Euclid".

The book is dedicated to the boys who could master the fourth proposition on congruent triangles. They were called *ἀι μωταί*, or the Initiated, and their names were placed on a list in his classroom. This title was also affixed to their names in the published mathematical sets, and it was interesting to find that in a recently published book called *Changing Eton* two eminent Etonians confessed themselves puzzled by the use of this phrase in old school lists. So quickly does the memory of these things disappear! But Hawtrey and others failed to discover the reason why boys found their Euclid so difficult. They would not or could not realise that to a boy logical deduction is difficult; whereas to an adult it is far easier. Some of us must remember how as boys of eleven and twelve such difficulties came to us, and surely these recollections have always helped us in our teaching days? But the teachers of those days were unrealistic in their approach, and always steadily insisted in writing and conversation on the value of Euclid in teaching logic. Hawtrey sets virtually no riders in his book, but pages of questions about the euclidean propositions of a childlike character, such as: "What is *AB*." Answer, "a line", and so on. Before I leave him, may I quote a paragraph from his *Narrative Essay on a Liberal Education*. It shows how an enthusiastic teacher can deceive himself. "For myself", he says, "I can remember few hours of greater enjoyment than those in which, taking up a book which has held its ground for more than 2000 years I have endeavoured by its help to open the minds of the young. . . . Look into those boys' faces! They are merry hearted. Look at them now. Mark the fixed eye, the riveted attention, the smile of satisfaction. What is this all the sign of? The interest and pleasure felt in the exercise of their mental faculties in following out intellectual truth. Give me your sympathy, my friend, and tell me if you do not think that Euclid is doing them a great good." And so he goes on with similar rhapsodies, and sincere but mistaken enthusiasm.

The development at Harrow was on rather different lines. I have spoken of Marillier who came on a voluntary basis in 1819, and was at the school for nearly fifty years. But the first real advance occurred when Colenso (2nd Wrangler in 1836 and again a Johnian) was appointed to join him in 1838. Unfortunately he was only there for four years, due no doubt to the fact that it was a bad time in the school's famous history. It was Vaughan who rebuilt it in 1845, three years after Colenso had gone. Indeed the Vicar of Harrow, a Governor of the school, advised him in 1845 to expel all the school and have some back on his own conditions. If Colenso had come ten years later he might not have left, and trouble in Natal might have been averted. His well-known *Arithmetic*, and perhaps his *Algebra*, were famous textbooks in their day. Indeed I have already referred to the former. He became really well known as Bishop of Natal. Theologically he was in advance of his time, and his writings caused him to be tried for heresy. The story is long and tortuous, and its effects were still evident in Natal when I lived there some twenty years ago quite close to his two surviving daughters. I like to think that he made his impression during the four years he was at Harrow. At all events his book

was widely used and replaced Cocker and others. Hundreds of schools were using it in the 'sixties of last century.

Middlemist, who replaced Colenso, was there for many years, and about him many stories were told. He was a very strict disciplinarian and, like Hawtreys at Eton, insisted upon a perfect rendering of Euclid. One of his colleagues, in despair at failing to make his boys understand the fifth and other propositions, had figures with triangles coloured where they overlapped, and boys frightened by Middlemist used to go to him in the hope that his, so-called, "new way" would enable them to understand, and save a detention or worse. Years ago I was shown some of these early models which dated from the 'sixties of last century. No doubt they were primitive, and they have not survived. One or two other models that have survived from those times are on view in an adjoining room.

The author of these early euclidean models lodged with a Mrs. Wood, and, distressed by the obtuseness of the boys, used to take Euclid with these coloured diagrams to the young and intelligent Miss Wood to see if such figures helped her to understand. This Miss Wood became Mrs. Annie Besant, the well known Indian theosophist.

Other stories were told of Middlemist as they were told of Hawtreys. He was a house master and rigid as regards punctuality. He actually once refused admission to two boys who were a few minutes late for "lock up" and they were found half an hour later by another colleague in the local hotel seeking accommodation for the night. He lived alone and was reputed to be a bachelor. Great was the amazement of the local society when, on his death, it was discovered that he had left his money to his widow and children living in a south coast town. It seems incredible that this should never have been known during his lifetime.

The Harrow mathematical staff of the 'sixties showed an advance to four. There were no less than three Fellows of the Royal Society at the school, but one of them, Farrar, though a man of wide interests, was not a mathematician. He was afterwards Master of Marlborough and Dean of Canterbury. Hayward, for ten years President of this Association, was, perhaps, the best of the other two as a student, and had done original work in the use of moving axes. It ought not to be forgotten that this mathematical staff was responsible for introducing the range system of marking, with a fixed minimum, as well as a fixed maximum, and in 1875 published a small brochure on marks headed by a Shakespearean quotation :

Ghost : Mark me.

Hamlet : I will.

Archaic systems of marking have lasted for many years, certainly for an appreciable part of my teaching career. Under many of them the weekly placing of boys could not possibly be regarded as fair, equitable or just. To the Harrow mathematical staff of three-quarters of a century ago must be given priority in devising a fairly just range system, which is to-day widely used. Of the actual teaching I know something derived from actual discussions forty years ago with masters who were teaching forty years before that. Further, to the end of his life, which occurred during the last war, at the age of over ninety, I kept in close touch with Gerald Rendall, my old head-master. He was a boy at Harrow from 1864 to 1870 where his father was a house master, and, though a prominent Classic, attained to the highest honours in Mathematics. He always regarded this period as a time when, in Mathematics, the old was giving place to the new ; the older mathematical masters were giving way to a younger and better trained generation of teachers. Rendall, and a few like him, would do some Trigonometry, Mechanics and

Conic Sections but rarely any Calculus. It was, however, only the very few that went as far as this; the vast majority of boys did nothing beyond Arithmetic, Algebra and Euclid and not always did they learn his sixth book on similarity.

I have stated it was Euclid that dominated the scene. I have a note-book containing agenda from mathematical masters' meetings at a well-known school, belonging to the last century. There are continual discussions as to what is to be done to those boys who fail to reproduce the book work. Regular rules are laid down about detentions. Boys who fail have to do further papers containing questions on the propositions in the various books, and on the Definitions, Axioms and Postulates. But no mention is made of riders. It would appear that boys might be forgiven for an inability in other mathematical work, but Euclid had gained such a position that accuracy, almost verbal accuracy, was demanded. I imagine that my own experiences at school in the 'nineties were not dissimilar to those in the 'sixties. I have a vivid recollection to-day of my introduction to the Fifth Proposition, or Pons Asinorum, at the age of eleven. Euclid's ungainly construction was due, as I might remind you, to the fact that the hypothetical construction, as it was called, of bisecting the vertical angle, when the pupil had not learnt how to do it, was not permitted. There is a famous school song called "Euclid" dealing with this proposition, and I am glad to know it is still sometimes sung at the school concerts. Perhaps my main memory of it is associated with thought-reading. It was about forty or more years ago that Mr. and Mrs. Zancig startled England with their powers. Mrs. Zancig stood on the platform with her back to the audience, while her husband willed her to give the name of objects shown him as he walked round the room. A professorial Fellow of my college, Sir Robert Ball, the eminent astronomer and a former President of our Association, was sitting close to me in the Guildhall at Cambridge. He produced a large figure of the Pons Asinorum. Mr. Zancig turned it upside down and said "What is this?" and appeared never to have seen the figure before. "Euclid", replied Sir Robert. And Mr. Zancig then compelled his partner, by thought-transference, in a moment of time, to draw it on the board upside down as he was holding it. I have been a believer in thought-reading ever since that date.

I suppose it is fair to say that most boys did not understand properly many geometrical ideas. A common practice in a class of forty at the beginning of this century was to say Euclid in school during that frigid hour before breakfast, each boy in succession saying one sentence with verbal accuracy. For instance, a boy got into serious trouble if, in the fourth proposition, he omitted the famous "each to each" which was regarded as essential: "If two sides of a triangle are equal to two sides of another, each to each." I could relate a score of such absurdities and, remember, although I am speaking of a period in the 'nineties shortly before the deposition of Euclid, I would reiterate that there had been virtually no change in the form of teaching over a long period. I suppose in order of popularity Euclid was then at the bottom of the poll: now, in the hands of a good teacher, Geometry must surely be almost at the top. It is because deduction is not suited to a beginner. Induction must come first.

We all know that Arithmetic had not gone far in Alexandrian days. Euclid almost entirely avoided it in the books read in England by schoolboys, and it is a strange historical fact that the divorce should have lasted till the twentieth century. Further, the survival of Book II was curious. It is mainly Algebra in geometrical form and, during my teaching days, a few of these propositions have survived as illustrating algebraical identities, but I am not sure they are going to survive much longer save as mere geometrical illustrations of factors.

But I have not quite finished with Euclid yet. I might say a word or two about Rugby to which our former President, Canon J. M. Wilson, went as a

master in 1859. He was a great schoolmaster, and subsequently a great headmaster. He is more famous for the impetus he gave to Science teaching, practically starting it on a systematic basis at Rugby. He commenced teaching in the cloak room of the Town Hall, but in 1861 a new school was built for Chemistry and Science. His idea of starting with Astronomy, Geology and Botany, as likely to give the greatest stimulus at an early age, has never, however, taken root in schools.

But he soon became seriously perturbed about his Euclid teaching. He tells us that he got no sympathy from his colleagues, but perceived clearly that it did not bring out originality of thought, and was disliked by the boys. Also the knowledge gained was slight, as after two or three years the boys only knew imperfectly two or three books of Euclid. So he gave them easy exercises, and put before them the idea that the aim of the first two books was to make a square equal to any area bounded by straight lines. To this the boys responded nobly, and there was a considerable advance. No doubt his own personality was largely responsible. Then came the Schools Enquiry Commission of the late 'sixties. Temple, his famous headmaster, was a leading member, and asked Wilson to ascertain why Geometry was so backward in English schools compared to those of Europe, and whether it was due to the retention of Euclid. According to Wilson he then made the discovery, new to him, that the essential difference was that in Euclidean Geometry no hypothetical constructions were allowed. Or, in plain English, you must not assume a line has a middle point, or an angle a bisector, until, with a straight-edge and a pair of compasses, you can find the bisecting point or line and prove they are such. This rule had been abandoned on the continent, but not in England.

Actually hypothetical constructions are the very first thing put down for consideration at the first meeting of our Association on 17th January, 1871. But it was thirty years before they were allowed. This caused Wilson to draft a new syllabus which led to strong opposition from many, including De Morgan, Todhunter and, it is interesting to note, Lewis Carroll. Indeed, in 1879, the latter publishes an elaborate defence of Euclid under his real name C. L. Dodgson, with an argument conducted in dramatic form. Minos, Euclid and others appear; Nostradamus is a defendant; the word meaning "we give quack remedies". Wilson, his elementary Geometry and our Association, then called the A.I.G.T., are arraigned and condemned. No doubt Dodgson always had Lewis Carroll close behind him. De Morgan, an ally, in his review of Wilson's *Geometry* says, "We feel confident that no such system as Mr. Wilson has put forward will replace Euclid in this country".

However, some presumably thought otherwise, because when the first meeting of the A.I.G.T. takes place in 1871 in Wilson's house at Rugby, Professor Hirst is elected President, and Wilson one of the two Vice-Presidents.

And what about the examination papers during this period? I have examined those set from the 'forties onwards in at all events one famous school. From 1850 to 1900 little change occurred; far the duller paper is what is called Euclid. Nearly all the questions are propositions, and it must not be forgotten that they had to be answered in euclidean language. An occasional rider occurs but all the constructions are euclidean, and no definite angle, such as 40° , is ever named. It is always "the given angle". Right up to the reforms of 1900 it is the propositions that matter.

I do not doubt that Euclid was a great discoverer, but, I must reiterate again, that it is tragic for those of us who believe in the world's destiny to think that man's intellect and spirit had got so dimmed, that the letter had taken the place of the spirit, and that 2000 years later we were still learning what had been propounded originally in the schools of Alexandria.

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The Arithmetic and Algebra changed perhaps a little more in those examination papers over fifty years. Cube root, for instance, had disappeared by 1900, but the mysteries of recurring decimals survived in all their complexity. Fractions are more valued than decimals, and there was an absurd enthusiasm for the usual complicated fractions in both Arithmetic and Algebra. But it was not all bad; some of the questions set in 1850 could be set, and indeed are being set, to-day.

So far a few schools have been mentioned. A good deal more could be said about their mathematical work, as it can be gleaned from the pages of those two massive volumes, called the *Report on the Management of the nine Public Schools*, published in 1864. It is sometimes called the Clarendon Commission. They are worth reading to-day; but this Association has already in a previous paper heard something of what is written in those pages and I hesitate to say more. There are no fundamental differences between the mathematical work at most of them at that date. Roughly Mathematics started at those nine schools as a voluntary subject at the beginning of the nineteenth century; became compulsory a little later. The work was dominated by Euclid till 1900, and even the other elementary mathematical subjects were rather dull, stodgy and formal.

But even more interesting, though less well known, is the Schools Enquiry, or Taunton, Commission of 1865. It is sometimes called the Endowed Schools Commission, though it dealt with private schools as well. No less than 782 endowed schools are concerned. Some date from mediæval times and some are of later foundation.

The national schools for the, so-called, labouring classes are excluded, but all others for what are called the middle classes are included. We in England have inherited many things from the past. I should give priority to our parish churches, but second only to them are our endowed schools, the majority of which to-day are absorbed in the local system. This Commission, however, enquired not only into the 782 endowed schools, or grammar schools, but also into the comparatively few proprietary schools owned by individuals or group of individuals but not by the headmaster, and into private schools, actually owned by the headmaster. It is striking to discover that they estimate the number of private schools as 10,000; exactly the same estimate was made by the Private Schools Commission of fifteen years ago. This report naturally gives a much truer account of mathematical teaching in the schools than the one restricted to the nine. A good deal can be ascertained from the twenty-one massive volumes of the Report issued in 1868. Different Commissioners dealt with different counties, so that opinions expressed vary somewhat. With twenty-one volumes, each of great length, the merest sketch only can be given. Mathematical textbooks are generally mentioned. The favourites, as I have stated, are by Colenso and Todhunter. These are clearly the commonest books used in Arithmetic, Algebra and Euclid. Euclid is regarded in much the same way as Latin as a necessary element in education. The knowledge of Euclid is considered much inferior to that shown in Arithmetic. "Boys scarcely seemed to know the first twenty propositions" is the kind of remark we read. But the Arithmetic is criticised as well. Boys were apt to regard figures as mysterious signs with fixed laws and rules which had to be committed to memory, but had no bearing on everyday life. In short, the criticism is similar to that of De Morgan in his *Arithmetic* half a century earlier, who insisted on the importance of understanding principles rather than rules. I suppose we give much the same criticism to-day. But it is recognised that many of the schools were languishing, and they went on languishing till the Balfour Act of 1902 started the great awakening.

It is interesting to note that private schools were in general very unwilling

to answer and fill up questionnaires. In Sussex and Surrey, for instance, out of 373 private schools circularised only 110 responded, and similar percentages are given for other districts. Parliament insisted on the inspection of the endowed schools because their position under the Charitable Trust Acts of 1853 made them more dependent on authority. But the private schools were in a position to reject the Commissioner's visit, and often did so.

The mathematical standard, however, in private schools is often negligible; obviously the masters were of very poor quality, and scarcely knew any Mathematics themselves. There are plenty of gems. One master is reported to be a good teacher of Euclid when he was sober. This in a district, and it is not unique, where the average residential salary was £64 per annum. The good teacher of Arithmetic is often reported as out of his depth when he teaches Algebra, and it is not uncommon to find few boys of fifteen who have reached, in Algebra, as far as simple equations. But even they knew little, and they looked on Algebra as something quite different from Arithmetic. Girls' schools are worse. Generally they did little but Arithmetic, and at the few schools where the Commissioner found them proficient in the first few propositions of Euclid they were generally taught by a visiting master.

But it must not be supposed that this ignorance was universal. There are certain star schools in every district. In the London area, for instance, with a population of three millions, the Inspector particularly mentions those two great schools, the City of London and Christ's Hospital. Only two or three boys at one of the nine public schools would have done as much as ten or twelve at these two schools. Work was being done on all the usual subjects taught in our top forms to-day, and it is interesting that the Integral as well as the Differential Calculus is included. Hawtrey at Eton had never taught even the Differential. It is well known that the City of London School produced more than one Senior Wrangler about this time.

In the Midlands, however, the prospect is bleak. Even at King Edward's Birmingham, only two boys above Form IV did extra Mathematics instead of verses. Clearly the time of Rawdon Levett had not yet come. The Commissioner is indignant that Mathematics did not count for Leaving Exhibitions. Of all the endowed schools in this area there were only five in which anything beyond Arithmetic, Algebra and Euclid was taught; and only twelve boys in these five were doing Trigonometry.

Or Norfolk might be taken, where another Inspector finds twenty endowed schools and large numbers of private schools. Yet only three had got as far as Trigonometry and then with but a boy or two, and no Algebra except at Norwich extended beyond the solution of quadratic equations.

Further south in Suffolk, we read that the Euclid done is worthless, but at the best known school the senior boys have some knowledge up to the middle of the First Book. Then to the North Midlands. Even at Uppingham, the mathematical standard is low, though Thring had been there for some years. One reason assigned is that the whole school is engaged in Mathematics at the same time, and all the form masters take part in the teaching. One of Thring's fundamental rules that no master shall teach too large a form seems to be disregarded in Mathematics. Further, good work does not affect a boy's position in form. I might add that the criticism that the whole school does Mathematics at the same time occurs elsewhere, even at Isaac Newton's school at Grantham, though both the headmaster and second master were mathematicians. I have known this happen in my early days of teaching at quite well-known schools, in which case, needless to say, the mathematical results were poor. Then on to the Lincolnshire area, where again we read that the general standard in Mathematics is low. In the north-east the picture is worse; the masters themselves knew very little; and only the most elementary

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Arithmetic is done, and even that in accordance with a set of unexplained rules. There is very little Euclid, but riders are almost unknown. Indeed, at one school the Commissioner suggests, as a reason, that the master was quite unable to do them himself.

There are many criticisms of the lack of equipment. Sometimes there are no blackboards, so that the master had to work out his problems on paper and then turn it round in order that the boys clustering round his desk could read it.

In their summing up, the report is very critical even of the endowed schools. "The teaching of Mathematics is rarely satisfactory." "The teachers do not take much interest." "The methods of teaching want improvement", and then comes the query "Whether Euclid be the proper text book for beginners, and whether boys should not commence with something easier and less abstract". Further the Commission, even at this date, suggest that the practical application should precede the theoretical; and that, if properly taught, the results attained would soon prove their value. This recommendation, in particular, excites the anger of Lewis Carroll, Todhunter and others. Probably it was written by Temple, a member of the Commission, who had been influenced by his colleague, J. M. Wilson, at Rugby. It is supported by Mr. Griffith, the Secretary to the British Association, who is an opponent of Euclid; later he became the first Science master at Harrow in 1867 and was there for twenty-six years.

The Mathematics of private schools is said to be worse. They place Arithmetic in a more favoured position, but the results are not superior to those at Grammar Schools, while Mathematics in general is much inferior.

At girls' schools the report is even more emphatic about the unsatisfactory character of mathematical teaching.

The picture is quite clear. A very small amount of Mathematics, easy Arithmetic, some earlier propositions in Euclid, and perhaps the very elements of Algebra are being attempted; very occasionally some Trigonometry. In the better schools, no doubt the few boys were going further, but in general masters could not cope themselves with anything but the elements. We must remember this report is issued in 1868, and our Association started only a year or two later. Surely the most pessimistic among us must feel that the enormous advance in three-quarters of a century is largely due to the efforts of our predecessors in this Association. We need remember only the tens of thousands of the young who are doing Trigonometry and more to-day.

And what was the reason for this low standard? Largely, the almost purely classical curriculum. It is worth examining the time-table of a famous school in 1600. Arithmetic is done by the two lower forms on Saturday afternoon from 2-5, together with the Church catechism, and there was no real change till a century ago. Then other subjects crept in, and were gradually accepted, headed by Mathematics, but even in my schooldays, Mathematics, French, Science and English subjects were but the Cinderellas of the time-table, and it was often open to the able Classics to avoid them. Up to the last year but one of my schooldays I had to produce four classical compositions weekly, two in prose and two in verse in two different languages. The so-called overcrowded time-table of to-day is of course merely due to the fact that the teachers of all subjects now exact their pound of flesh in a way that was not done fifty years ago.

Yet many people were clamouring for reform. When the Public School Commission reported, distinguished men entered into the fray. In 1867, Farrar, the well-known Harrow master, delivers what is in its day a famous lecture before the Royal Institution which he calls "Some Defects in Public School Education". He rebels against what he terms the "Infructuous years of Classical verse making". There is universal agreement, including a leading

article in *The Times*, that public school education is a failure. On the purely mathematical side, Dr. Whewell, the great Master of Trinity, offers his views, and reiterates those he published as early as 1845 called *A Liberal Education*. He claims, and had claimed in 1845, that Practical Mensuration and Logarithms should be taught; in short he wants concrete Mathematics at school and the theory at the university. He, like many another, deprecates the continuance of universal classical verse making. He denounces the neglect of Arithmetic, and states that many boys who have received a good education are ignorant and helpless (to use his words) in numerical calculation. Hence he suggests that a sound mathematical education at a later date is almost impossible. All he asks is that boys should have a thorough knowledge of Arithmetic. Algebra and Geometry can then be taught, in a more deductive sense, at the university. He does not mind an extension to Euclid, Algebra and Trigonometry, but he insists at all costs on a thorough knowledge of Arithmetic first. All this does not appear to be asking for much. He asks it in 1845, and again in 1865 in his statement to the Public Schools Commission.

Sir John Herschel, the astronomer, is more ambitious; he wants the upper forms to do plane and spherical Trigonometry, Conic Sections and higher Algebra and Mechanics. He certainly implies that a substantial number of boys should carry this out at all schools which keep their boys till eighteen.

So with others; but I doubt if there was much difference till the end of the century. The plain fact was that schools would not spare sufficient time from the classical studies, or else the least satisfactory periods were devoted to the "Cinderella" subjects. In Mr. Hoyland's recent book on Uppingham, for instance, he speaks of "certain afternoon periods in the 'sixties being devoted to Mathematics, and as Thring himself took no part in this subject, and the masters concerned were not very efficient, the periods seem to have been regarded as a drowsy opportunity for the peaceful digestion of the school beef. Story books were concealed behind desks, there was a good deal of name carving and the barest minimum of work." Actually three afternoons a week were devoted in lower forms to Mathematics, and very often one of them was an extra half holiday. Occasional good mathematicians at the top had to do work out of school as an extra. Canon Wilson, too, talks in his autobiography of the Rugby time-table in 1859. He took the second set in the sixth form for three periods a week and he adds "no one wished to learn and they were all working at something else". In the fifth form he takes two sets, again in the cloak room of the Town Hall, a room completely unsuited for class teaching. The class was dull, bored and uninterested, and failed to bring any preparation. On a further enquiry they informed him that it was an immemorial custom to do no preparation in Mathematics as long as they behaved well in form. Wilson claims that he was not a good mathematical master, but those who knew him well scarcely credit that. At all events he adds that he did not know what good mathematical teaching was till he went to Clifton as headmaster in 1879, and saw the work of Hall and Stevens.

The late Provost of Eton's testimony is much the same. Here is his record of a mathematical lesson at Eton about 1880.

Starting at 5 p.m. when the best of the day was over, the form meets, but it was understood that the selection of the work was to rest with themselves. "What are you going to do, X?" "I was thinking of doing a little Algebra this afternoon, Sir." "Very well." There was no crude demand for results, and a deep tranquillity pervaded the classroom. Exactly the same could happen still in my day at school. In the upper forms it was generally understood that Latin verses might be substituted for Algebra provided it was not

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done too openly or too blatantly. Mathematics was indeed, often, even in 1900, a "Cinderella" subject.

And now as regards the times of change. Mathematical reform marched arm-in-arm with educational reform. The Act of 1870 is the first national landmark as it made education practically compulsory, or at all events easily obtainable. The same year, or January 1871 if you prefer, saw the birth of our Association for the Improvement of Geometrical Teaching. The year 1902 saw the Balfour Act, and mathematically coincided with the dethronement of Euclid. How great a reform the latter was, only those who suffered under Euclid can probably realise. The year 1917 saw the Fisher Act, and 1944 the Butler Act. We note that all these three latter Acts came after a great war, and may well be due to a kind of unconscious belief that, after the horrors of war, all, in expiation, were resolved that the next generation should have a better deal. Will not future mathematical historians say that the Butler Act coincided too, as well as the others, with another period of mathematical reform? It seems to me that everything points that way. It is due to a variety of circumstances. We all know, for instance, the importance attached to Mathematics by the R.A.F. which expanded to so enormous a figure during the war. Mathematics had to be taught to tens of thousands in the A.T.C. A greater number of technicians are required, all of whom demand some knowledge of the subject. The old formalities started to vanish some forty years ago, and many have gone, but the ground is fertile for further reforms, such as the combination of Geometry and Trigonometry, and an earlier entry of the Calculus. Further, owing to a rise in the school age, millions will learn more than their fathers, and on our Association must fall some responsibility for suggestions as to the syllabus that will be required. Hence I suggest that we are on the eve of another great mathematical advance, marching in line with the Butler Act.

Thus 1870 is a great date for us, and for education in our country. So is 1902. And so I venture to think may be 1944. Three great landmarks in our educational history and three great landmarks in the history of our Association; both of the last two at the conclusion of an exhausting war, which seems to make changes and alterations so much more possible in the hearts and minds of men.

Now about these reform periods. Much is known to you. It might, however, be said that it is somewhat surprising that so little was really done to reform Mathematics between 1870 and 1900 in spite of the criticisms of the two Commissions, and in spite of the activities of our Association. There was certainly considerable reform in the government of the schools, due to the recommendations of the Commissioners, but not in the curriculum, and for this there are two main reasons. Partly because the universities retained Euclid and formal Algebra, and partly because of the complacency of the schools. It was a time of great prosperity, especially for the better known schools, and they tended to become more conventional and stereotyped. The smaller endowed schools followed suit. Everything was good in a perfect world, and even complicated arithmetical and algebraical fractions had made us what we were, while long trigonometrical identities were somehow bound up with our imperial destiny. That was the kind of conservative theory of the nineteenth century, and hence change was difficult. But discontented spirits have always existed. Murmurs grew until, a little over forty years ago, Oxford was the first to accept the reform proposals, though it was said with an uncertain voice. Cambridge moved a little later, but with more decision, and in 1904 held its first Entrance Examination without insisting on Euclid. The schools altered at once, thus showing once again how absolutely English school teaching is governed by examinations, and thus ultimately by the universities.

When the universities moved, reform came upon the schools like a flood, and a flood it was. But the ground had been prepared, and the flood was led, if a flood can be led, by able men.

I have always felt that a debt is owing to the mathematical examiners of the Civil Service Commissioners. Even to-day it is worth while looking at the papers they set forty years ago, and they might well be set to-day. Further, I would number among the leaders of the revolt, and the guides in the new Mathematics, W. C. Fletcher, a Johnian again, who, as Senior Inspector, was responsible for a number of the pamphlets issued by the Board, and whose inspiration many will gratefully remember.

Then there was that very wise counsellor and trainer of teachers, Sir Percy Nunn, whose influence was as great as his wisdom. And lastly, that remarkable partnership of Godfrey and Siddons, who, as spiritual disciples of Levett, the virtual founder of our Association, provided those earlier reformed textbooks, which were a guide to so many, and who, as counsellors in committees, did work on a level which has rarely been surpassed.

I have said reform came like a flood. Indeed it did. In the year 1900 the Board of Education had issued a report about mathematical teaching at preparatory schools. We are told, as late as 1900, that by the majority of preparatory schoolboys three books of Euclid can be managed, while the more mathematically inclined will add the fourth and sixth without much trouble. The wording of the text must be insisted on. They should also do a substantial amount of Algebra and Arithmetic. We are further informed that two lessons weekly of fifty minutes or one hour will be ample for this. I repeat that the above was written as late as 1900. How completely different from the report on the teaching of Mathematics in preparatory schools issued by our Association in 1907!

It is astonishing that two such documents dealing with the same subject at similar schools can be written by authoritative bodies at an interval of seven years and be so completely dissimilar. I might add that it is surprising that the 1907 report is so good, as there were no less than thirty-five members on the Committee! Incidentally, that Committee included Mr. Siddons and Mr. Tuckey who are active on our councils to-day!

This was not the earliest report, for I might almost say that reports, as well as reform, came like a flood, and very good they generally were. Not only did the Mathematical Association get busy, but syndicates at Oxford and Cambridge were busy too, their main contribution being, of course, the abandonment of Euclid. Further, the appropriate sections of the British Association listened to the voice of Professor Perry at the meeting at Glasgow in 1901, and in 1902 at Belfast both he and Mr. Eggar of Eton brought forward suggested mathematical syllabuses for schools. To the supporters of Euclid they argued that there was plenty of logical chaos in Geometrical Conics. Why not in Euclid? They insisted on the association of Arithmetic and Algebra with Geometry especially in Book II of Euclid. They were adamant that no candidate must pass a Geometry examination without showing some facility in riders. Indeed, they demanded a proper balance between experimental work, didactic teaching and numerical exercise work. And then, of course, they added the usual warnings against elaborate manipulations, indices, brackets, fractions which were so common in those days.

But there were doubters. We have all smiled over the story of Cayley, the great Cambridge mathematician, who refused to allow Euclid to be abolished as a subject for the Entrance Examination to the university. He is said to have stated that the proper way to learn Geometry was to start with that of n dimensions, and subsequently to come down to the special cases when n is equal to 2 or 3. Cayley, of course, was out of touch with realities; and no

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doubt it was extremely difficult for Cambridge to move without his consent, which he was far too conservative to give. He was alleged rarely to recognise anyone in the streets of Cambridge as he was always gazing into infinity! His peculiar walk made him a familiar figure there.

His great contemporary was Sir George Stokes, who worked chiefly at Mathematical Physics, while Cayley's immense researches were in pure Mathematics. Actually nine hundred papers of the latter have been published in thirteen volumes. Stokes does not seem to have entered much into the fray, though he survived Cayley by several years. The provost of my college told me of his only encounter with Stokes at a sexcentenary dinner at Peterhouse in 1884 where Lord Kelvin had just installed electric light. He sat next to Stokes, who smiled brilliantly but said not a word to anyone. At the conclusion he was taken by Stokes and Kelvin to see how the electric light was working, and Stokes, as he closed the door, again smiled brilliantly and said in a low but intense voice "We won't go home till morning". This, added the provost, was about the only observation, grave or gay, that I ever heard from his lips.

We do not meet with so many strange creatures among the dons to-day. I imagine it is mainly due to the abolition of celibacy, which is comparatively recent. The next generation of Cambridge dons were more understanding of human frailty, and we owe a debt to men like Forsyth and Hobson who supported the abolition of Euclid from the entrance examination.

That was the obvious preliminary condition for reform. My only personal recollection is the attitude of E. W. Hobson, with whom I once journeyed across Canada when we were both attending in 1909 a meeting of the British Association at Winnipeg. He gave me his views about the teaching of Mathematics in schools, and voiced them a year later as President of Section A at Sheffield. But even in 1909 he had ideas which he desired to bring out in his Presidential Address. The prophets of evil had always stated that the abandonment of Euclid's order made the task of the examiner impossible. Hobson freely admitted it made it harder, but utterly denied that you must help the examiner at the expense of the pupil. He specially spoke to me of Geometry and Mechanics. In the traditional treatment both deduction and induction occurred, but far too much stress was laid on the former in the early stages. These proportions should be reversed, and in the earliest stages they should be treated almost entirely as observational studies. I remember this and much else that he told me. The curious can find it in the British Association report for 1910; but I think a tribute should once again be paid to a great friend and supporter of mathematical reform. His views on Mechanics are interesting. We must remember that the practical teaching of Mechanics is a modern growth. In 1909 the report made by a Joint Committee of the Mathematical Association and the Public Schools Science Masters states: "In no subject is the want of co-operation between the Mathematical and Science Masters so apparent as in Mechanics. In the large majority of schools, Mechanics is taught by the mathematical master as a part of the regular mathematical work; no experiments are made, and in very few cases are practical difficulties mentioned. . . . There is no connection between the lessons given by the two masters." I doubt if the cooperation to-day, forty years later, is always as close as desirable. The ideal that the mathematical master should teach both the practical and the theoretical is, I know, often administratively difficult; it is, however, the ideal to aim at.

But there were other doubters. My period at the university coincided with the beginning of the period of reform. My mathematical tutor, a Senior Wrangler, was not wholly sympathetic. He was particularly critical about the teaching of graphs. I daresay a good deal of it in early days was rather

aimless, but we were put right by the Board of Education's pamphlet of 1907, mainly written by Fletcher, which urged strongly that we were treating them from the point of view of Analytical Geometry, whereas we ought to be treating them from the point of view of the Calculus. It is unfortunate that even to-day, in many examinations, the Analytical Geometry point of view so often prevails. Probably it is easier to frame such questions. However, this mathematical tutor asked me what we were doing, and asserted that when, after two lessons, a boy had learnt two graphs there was nothing more for him to do. He was thinking, I presume, of the parabola and hyperbola in their simpler form.

Another tutor, and mathematician of high calibre, was particularly concerned about the teaching of Mechanics. He claimed that it lacked principles in its elementary stages, and was afraid of the practical side taking the place of the theoretical. He argued that all boys ought to commence with the principle of virtual work, and hoped that when my teaching commenced I would write a book on that basis. Wisely and prudently I declined.

And there are other points about these reforms. It was high time they came. There was an increasing demand for Mathematics, but not for the Mathematics taught prior to 1900. It was for Mathematics with a practical bias rather than for a collection of dull, meaningless symbols. Charles Rolls, the great motor engineer, whose statue is, I think, the glory of the town of Monmouth, said "What is the use of talking about $\cos \theta$ if you cannot use a spanner?" Most of us perhaps would prefer to go up in an aeroplane with a man who could use a spanner rather than with a theoretical mathematician, but, on the other hand, I do not suppose the aeroplane or engine would have ever been there if some of us had not known about $\cos \theta$. Cooperation, however, is necessary. Mathematics cannot divorce itself from technicians, and the last forty years has seen an increasingly closer collaboration. No boy now in his school Mathematics can avoid laboratory work like my generation did. But an intensive knowledge of Mathematics is less common.

In this brief survey I do find it necessary to remind you that in the last examination of the Northern Universities Joint Board for Higher Certificates—the largest Board examining for Certificates in this country—over 3000 candidates offered Principal Mathematics, in general taking Principal Physics and/or Principal Chemistry as well; only twenty-four offered Mathematics as a single Principal subject of a standard approximating to that of open Scholarship. This compares very unfavourably with the statistics of forty years ago, and is due partly to the difficulty of the open Scholarship papers in Mathematics at the universities, and partly to the lack of schoolmasters of sufficient ability to teach. It is only fair to say that Scholarship examiners deny vigorously that the papers are harder than those of forty years ago, though they have changed their form.

But Science and Mathematics together lead to careers denied to the man who offers Mathematics only, and perhaps the fewer that take the higher standard is compensated for by the very much higher number who take a reasonable standard. It is necessary, however, to be careful in supporting the career standpoint, or I shall be reminded of Dean Gaisford's famous defence of Greek: "Nor can I do better in conclusion than impress upon you the study of Greek Literature, which not only elevates above the vulgar herd, but leads not infrequently to positions of considerable emolument."

There are scores to-day who have some knowledge of Mathematics compared to one who knew something a century, or even half a century, ago; but the number who reach the higher realms is less satisfactory.

Then something should be said as to the influence which changes in English mathematical teaching have had on English-speaking schools abroad. Schools

in the Dominions are the spiritual descendants of those in England. In early days they were largely staffed by men from this country who naturally brought with them the methods to which they were accustomed. That is why Science in Dominion schools became Physics and Chemistry, suited perhaps to an industrial country like England, but less suited to agricultural countries whose main interest lay in the biological direction. In the future, however, the Dominions are likely to rely more on their own men brought up at their own schools and universities, though it is hoped that the principle of exchange will continue, and indeed increase. From all points of view, including Mathematics, it is most important and desirable.

Some twenty years ago, however, I had the opportunity in Natal of preparing boys for the S. African Matriculation. In that country Scotch professors predominated, and I doubt if Scotland had moved as fast as England in mathematical reform. These professors were largely responsible for the matriculation examination to the universities, and sixth or post-matriculation forms were scarcely known. We, in England, are justly proud of the mathematical work done in these forms, and do not always realise that in other countries such work is generally done at the universities. Hence the matriculation tended to represent the summit of school mathematical teaching. It was, however, often taken about a year later than the age to which we are accustomed in England, and to those of us who valued the modern resilience and flexibility of our Examining Boards, the papers set there seemed to be of a rather dull and routine character.

In short, as you might expect, mathematical reform, as we see it, had not gone so far as with us. The textbooks in common use were mainly published in England and were those in use here a good many years earlier. A certain amount of manipulative skill was required, especially in Algebra, that we have largely discarded in elementary examinations. I had some connection with the formation of the Natal Mathematical and Scientific Society, and we urged certain changes, of which there is no time to speak, but it has taken time and naturally the war meant a certain cessation of activity. I believe there is still much to be done, though nothing can detract from the fine work in Education that has been achieved under difficulties in that country.

With events in Australia and New Zealand we are more familiar. Here, too, the schools are descendants of the English schools, having been largely staffed originally by Englishmen. But to-day there is a considerable reform movement. We are proud to know that there are several branches of our Association in that great country; indeed there is one in each of the states except Western Australia which, though the largest in area, has only a small population, and in Tasmania, the smallest state of all both in area and population. In New South Wales, sixteen members of their branch also belong to our parent Association, and therefore get copies of the *Gazette* direct, while over one hundred have copies of the *Gazette* circulated to them by the Secretaries. In 1945 arrangements were made in this state for the issue of a new journal called *The Australian Mathematical Teacher*, subscribers coming from every state of the Commonwealth, and from New Zealand. It has undoubtedly met a growing need. There is also considerable activity shown in forming a new syllabus in Mathematics for a Secondary School Certificate, particularly designed for the mass of pupils who do not want the traditional courses.

It is most stimulating to find this great country so alive to the mathematical needs of to-day, and it is particularly instructive and interesting to read all the aids to teachers that are being published to explain the changes, and to realise how closely they approximate to the principles we value so much here. It would seem that Australian Mathematics is in a very alive and satisfactory condition.

The development in Canada has been rather different. Although, especially in early stages, the alliance with English Mathematics was close, the influence of the United States has been naturally considerable. It is of interest, that unlike S. Africa, the mathematical textbooks are mainly published in Canada, and written by Canadians.

In America, 1875 is described as the epoch of change. In earlier days, owing to the War of Independence, France influenced the development of Mathematics, and in consequence the large French racial stock in Canada. But about 1875 the country had the good fortune to possess several great educationalists who became Presidents of universities. It was, too, at this time that J. J. Sylvester (2nd Wrangler in 1837 and again a Johnian) occupied a chair at one of their great universities for six years prior to his transference to the Savilian chair at Oxford. In an address in 1897 by Mr. Fabian Franklin, we read: "While there doubtless would in any case have been progress, it must be set down as pre-eminently the result of Sylvester's presence in Baltimore that Mathematical Science in America has received the remarkable impetus which the last twenty years has shown." I cannot resist quoting this great testimony to Sylvester as he was our President in 1891. The mathematical historian, Professor David Eugene Smith, who quotes the above, agrees that to Sylvester, more than to any other one man, is due the remarkable impetus given to Mathematics in America after 1875.

These general facts are worth mentioning, because a revolution in the universities must in due course come down to the schools. American school textbooks at this earlier date were similar to ours, and they paid little attention to the connection between Arithmetic, Algebra and Geometry. But change after 1875 was rapid. It is interesting to know that to-day the National Council of Mathematical Teachers, a school organisation, numbers over 4000 members, while there are two other advanced associations, mainly for university men, each numbering about 2000.

I have touched on the progress in America, because Canada has naturally been closely associated with, and shares the progress of, its southern neighbour. Probably we should regard their Mathematics as somewhat similar to that of S. Africa. In both countries the Scotch professor has considerable influence, and so, in consequence, the development as derived from the British Isles and apart from America has not gone on quite the same lines as here. For instance, the criticism is still made that the elementary subjects are kept in separate compartments, and a boy can have a pass in Geometry placed on his certificate without a corresponding pass in Algebra.

And what of the future? I myself should like to see a greater extension of mathematical laboratories. I suppose the expense of Science work is so considerable that one can hardly expect it. It is good for the mathematical master, as well as for his boys, that he should do some Practical Mensuration, Hydrostatics and Mechanics. It is a stimulating tonic for both. At the one school where I prevailed upon the authorities to supply it the success was considerable. Then there is that newcomer, the mathematical film, which should be an aid to teaching. We hear, too, more of models, and I believe in rough models made by the boys themselves. There is no finer method of pressing home a point. Nearly forty years ago, I produced a number to show at the International Congress of Mathematicians at Cambridge, and a few that have survived are in the exhibition in a neighbouring room. I beg you to regard them, not as belonging to to-day, but as relics from the past which have suffered over the course of years.

Then there is the correlation question. Thirty or more years ago this was giving considerable anxiety to teachers in Mathematics and Science, and a number of joint meetings were held between their professional bodies. I

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wonder whether the time is ripe for further consultations, even bringing in the Geographical staff?

But as I survey the history of mathematical teaching towards the end of my own career, one supreme fact comes to me which transcends all else. It is the efficiency of the teacher. Certainly he was not efficient seventy-five years ago. Things are improving, but they improve slowly. There are to-day far more efficient teachers. But no reform compares with the work of the enthusiastic and capable teacher. It is he, and only he, who can raise Mathematics to the height we desire.

And lastly, who can doubt that we are at a period of greater advance in England. A conference was convened in 1941 on the initiation of the Cambridge Local Examination Syndicate, when many fundamental changes were proposed. We are likely in the future to associate Geometry with Trigonometry to a far greater extent. I doubt if the writing out of geometrical propositions will continue for long. Calculus will become more and more prominent. Just as Euclid's Geometry was the high-water mark for the third century B.C., the Arabic notation in Europe for the twelfth A.D., the method of Long Division for the fifteenth, so perhaps the Calculus, the intellectual inheritance of the British race, will be the high-water mark for teaching in schools in the time in front of us. It will be a kind of waterway to the rich country ahead.

The value of all this is its stimulus. I suppose there must be some routine and rather dull work; but we must not emphasise it. Many a man has been heard to say he learnt nothing at school. But the point is whether the school stimulated him to teach himself. The dull routine of half a century or a century ago led the great mass of boys nowhere. Modern teaching by a skilled teacher can do anything. All of us need aids. I believe our Teaching Committee to be doing the most important work of the Association. The advent of the modern secondary school, with its desire for some new form or extension of its Mathematics, is a matter of great responsibility and opportunity. Our Association has not neglected its duty in the past. I do not myself doubt that it will successfully fulfil every claim and challenge upon it in the future.

W. F. B.

BACK NUMBERS OF THE GAZETTE.

Members who have copies of the *Gazette*, particularly of the scarce war-time numbers, for which they have no further use, are asked not to discard them, but to send them to the Editor.

THE AUSTRALIAN MATHEMATICS TEACHER.

The *Australian Mathematics Teacher* has now completed its second volume. Like the *Gazette*, it contains Articles, Notes, Reviews and Gleanings. In the present volume there are articles on: Vectors and Dual Vectors in Geometry, Formulæ for the triangles of Apollonius, Sixth Form Mathematics in New Zealand, End-of-the-Year Mathematics, The Arithmetic and Algebra of the Natural Numbers, The Logarithmic and Exponential Functions, Towards an Ideal Course in Geometry, Orthogonal Projection Methods in Trigonometry, Approximation, A Sequence in Trigonometry for Non-specialists.

The subscription is 5s. per annum, to be sent to the Business Manager, "Australian Mathematics Teacher", c/o Teachers' College, Sydney, New South Wales.

SOME NEW THEOREMS ON DIVISIBILITY.

BY R. L. GOODSTEIN AND M. RUMNEY.

Introduction. If we call the number $pa + qb$ the $T_{p,q}$ transform of the number $n = 10a + b$, then in general the $T_{p,q}$ transform of n will differ from n ; it is easily seen, however, that given any number n , we can always find a pair p, q such that the $T_{p,q}$ transform of n is n itself. In fact, if $n = 10a + b$ and $p = 10 - b$, $q = a + 1$ then the $T_{p,q}$ transform of n is $(10 - b)a + (a + 1)b = 10a + b = n$.

Let n be a number unchanged by a certain transformation $T_{p,q}$; what is the effect of the transformation $T_{p,q}$ upon any multiple of n ? Consider, for instance, the transformation $T_{2,7}$ and the number 34 which is unchanged by this transformation. Apply $T_{2,7}$ repeatedly to 12342 (which is a multiple of 34) and we obtain in turn the numbers $2.1234 + 7.2 = 2482$, $2.248 + 7.2 = 510$, $2.51 + 7.0 = 102$, $2.10 + 7.2 = 34$. Thus after four transformations 12342 is reduced to 34. Is this just an accident? Try the transformation $T_{3,5}$ in conjunction with the number 47 (which transforms into itself under $T_{3,5}$). Applying the transformation repeatedly to 10011, we have in turn $3.1001 + 5.1 = 3008$, $3.300 + 5.8 = 940$, $3.94 + 5.0 = 282$, $3.28 + 5.2 = 94$, $3.9 + 5.4 = 47$, and that again we reach the number unchanged by the transformation.

These examples suggest that if a number n is unchanged by a transformation $T_{p,q}$ then the repeated application of the transformation to any multiple of n brings us back to n . However, this is not the whole truth, for the process may terminate, not in n , but in a multiple of n which is also unchanged by the transformation; consider for instance multiples of 19 under the transformation $T_{1,2}$.

We shall show, amongst other results, that if N, N_1, N_2, \dots is the sequence obtained by the repeated application of a transformation $T_{p,q}$, and if n is a prime unchanged by the transformation, then N is divisible by n if the sequence N_r steadily decreases to a multiple of n which is unchanged by $T_{p,q}$, and N is not divisible by n if, at some stage, the sequence N_r increases. Thus the sequence of transformations affords a decisive test for divisibility. In practice the usefulness of the test is probably limited to those cases when the transformation may be effected mentally (e.g. $10^7 + 1$ under the transformation $T_{1,2}$, as a test for divisibility by 19).

Notation. The highest common factor of two integers p, q is denoted by (p, q) . If $p = kq$, $k \geq 0$, we write $p \mid q$, and if $p = kq$, $k \geq 2$, we write $p \parallel q$.

Every integer N admits a unique representation in the form $ma + b$, $m \geq 2$, $m > b$; if $(a, b) = 1$, then $N = ma + b$ is said to be an m -prime. If $ad - bc = 0$ then $ma + b$ is said to be m -similar to $mc + d$; it follows that if F is an m -prime, m -similar to N , then $N \mid F$.

We define the $T_{p,q}^m$ transform of $N = ma + b$ to be $pa + qb$ and write

$$T_{p,q}^m N = pa + qb.$$

The identical transformation $T_{m,1}^m$ is without interest and we suppose, throughout the paper, that $q > 1$. The letters F, N, p denote non-negative integers. If $T_{p,q}^m N = N$, then N is said to be an *invariant* of the transformation $T_{p,q}^m$. If $(p, q) = 1$, the transformation $T_{p,q}^m$ is said to be prime.

We prove a number of theorems on divisibility, all of them (except Th. 2) leading up to Th. 6.

Theorem 1. If $(p, q) = 1$ and $(pa + qb) \mid (pc + qd)$, then $(ad - bc) \mid (pc + qd)$.

i-01. In particular if $(ma + b) \mid (mc + d)$, then $(ad - bc) \mid (mc + d)$.

1.1. If $(p, q) = 1$, $(c, d) = 1$ and $(ad - bc) \mid (pc + qd)$, then
 $(pa + qb) \mid (pc + qd)$.

1.11. In particular if $(c, d) = 1$ and $(ad - bc) \mid (mc + d)$, then
 $(ma + b) \mid (mc + d)$.

Proof of 1. By hypothesis there is a k such that $pa + qb = (pc + qd)k$, and so
 $p(a - kc) = q(kd - b)$;

since $(p, q) = 1$, there is a ρ such that $a - kc = \rho q$, $kd - b = \rho p$, and therefore
 $ad - bc = kcd + qpd - kcd + ppc = \rho(pc + qd)$.

Proof of 1.1. There is a k such that $ad - bc = k(pc + qd)$, and so
 $(a - kq)d = (b + kp)c$;

since $(c, d) = 1$, there is an i such that $a - kq = ci$, $b + kp = di$, and therefore
 $pa + qb = kpcq + pci + qdi - kpcq = i(pc + qd)$.

Theorem 2. If $(c, d) = 1$ and $(pc + qd) \mid (mc + d)$, then $(qm - p) \mid (mc + d)$.

Proof. There is a k such that $(pc + qd) = (mc + d)k$, and so

$$(km - p)c = (q - k)d$$

hence there is a λ such that

$$km - p = \lambda d, \quad q - k = \lambda c \text{ and therefore } qm - p = \lambda(mc + d).$$

Theorem 3. F is an m -prime and an invariant of the transformation $T_{p,q}^m$.
 If N is also an invariant of $T_{p,q}^m$ then $N \mid F$.

Proof. If $N = ma + b$, $F = mc + d$, then

$$ma + b = pa + qb, \quad mc + d = pc + qd,$$

and so

$$(m - p)a = (q - 1)b, \quad (m - p)c = (q - 1)d;$$

but $(c, d) = 1$ and so there is a k such that

$$m - p = kd, \quad q - 1 = kc, \quad k \geq 1,$$

whence $ad - bc = 0$ and therefore $N \mid F$, by Th. 1.11.

Theorem 4. If F is an m -prime and an invariant of the transformation $T_{p,q}^m$,
 and if $N \mid F$, then $T_{p,q}^m N \mid F$.

4.1. Conversely, if $T_{p,q}^m N \mid F$, then $N \mid F$ provided the transformation is
 prime.

Proof. Let $N = ma + b$, $F = mc + d$; since F is an invariant m -prime

$$mc + d = pc + qd, \quad (c, d) = 1,$$

and therefore there is a j such that $m - p = jd$, $q - 1 = jc$, whence

$$pa + qb - (ma + b) = (p - m)a + (q - 1)b = j(bc - ad).$$

Hence if $(ma + b) \mid (mc + d)$ then, by Th. 1.01, $(bc - ad) \mid (mc + d)$

and so

$$(pa + qb) \mid (mc + d), \text{ which proves Th. 4.}$$

And if $(pa + qb) \mid (pc + qd)$ then, by Th. 1, $(bc - ad) \mid (pc + qd)$ and so

$$(ma + b) \mid (pc + qd), \text{ which proves 4.1.}$$

Theorem 5. F is an m -prime and an invariant of $T_{p,q}^m$. If N is m -similar to
 F then $N \mid F$ and N is an invariant of $T_{p,q}^m$; if N is not m -similar to F , and
 $N \nmid F$, then $T_{p,q}^m N < N$.

It follows that if $T_{p,q}^m N > N$, then N is not a multiple of F .

Proof. Write $N = ma + b$, $F = mc + d$. If N is m -similar to F , since F is an
 m -prime there is a k such that $a = ck$, $b = dk$, and so $N = kF$ and $pa + qb =$
 $k(pc + qd) = kF = N$. We consider next the more important case when N is
 not m -similar to F . Since $N \nmid F$, there is a k such that $ma + b = (mc + d)k$,
 $k > 1$, and so, if $dk = me + f$, $e \geq 0$, then $a = kc + e$, $b = f$.

Hence

$$\begin{aligned} N - T_{p,q}^m N &= ma + b - (pa + qb) \\ &= k(pc + qd) - (pkc + pe + qf) \\ &= q(me + f) - (pe + qf) \\ &= e(qm - p). \end{aligned}$$

But $e = a - ck$, $me = dk - b$; since N is not m -similar to F , therefore $a - ck$, $dk - b$ are not simultaneously zero, and so $e \geq 1$. Furthermore, since $pc + qd = mc + d$ and $q > 1$, therefore $p < m$ and so $qm - p > m - p \geq 1$. It follows that

$$N - T_{p,q}^m N > 1.$$

Theorem 6. F is an m -prime and an invariant of a prime transformation $T_{p,q}^m$, and $t_0 = N$, $t_{r+1} = T_{p,q}^m t_r$. If for some k , $t_{k+1} > t_k$, then N is not divisible by F . If $t_{r+1} < t_r$, $0 \leq r \leq \lambda - 1$, and $t_{\lambda+1} = t_\lambda$, then $N \mid F$.

Proof. If $t_{k+1} > t_k$, then by Th. 5, t_k is not divisible by F , and so, by Th. 4, $t_0 = N$ is not divisible by F .

If $t_{k+1} = t_k$, then, by Th. 3, $t_\lambda \mid F$ and so, by Th. 4-1, $t_0 \mid F$.

Observe that since p , q and N are positive, therefore all t_r are positive, and so a descending sequence t_0, t_1, t_2, \dots must terminate either with $t_{k+1} = t_k$ or with $t_{k+1} > t_k$ for some k . Accordingly one of the two possibilities considered in the theorem must arise, and Th. 6 is decisive.

Corresponding to any integer $F = mc + d$, we can determine positive integers p , q such that F is an invariant of $T_{p,q}^m$. For if $(c, d) = h$, $c = h\gamma$, $d = h\delta$, $(\gamma, \delta) = 1$, then $p\gamma + q\delta = m\gamma + \delta$ and so there is a k such that $p = m - k\delta$, $q = 1 + k\gamma$. Since $\delta \leq d < m$, we can determine p , q both positive such that $pc + qd = mc + d = F$, and so $p < F$, $q < F$.

It follows that if F is a positive prime, then $(c, d) = 1$, and there is a prime $T_{p,q}^m$ of which F is an invariant, for if $pc + qd$ is a prime F and $0 < p < F$, $0 < q < F$, then $(p, q) = 1$. Accordingly we can draw up a table of pairs p, q corresponding to primes F such that F is an invariant of a prime $T_{p,q}^m$. Then, by Th. 6, the repeated application of $T_{p,q}^m$ will determine whether any number N is divisible by F or not.

Examples. Starting with the number 8987 the repeated application of $T_{1,1}^{10}$ determines the sequence 912, 95, 19, the last of which is an invariant. Therefore 8987 is a multiple of 19.

Apply $T_{16,27}^{100}$ starting with 165066; we determine in turn 28182, 6710, 1342; since 1342 is an invariant of $T_{16,27}^{100}$, it follows that 165066 is a multiple of 1342.

The prime 23 is an invariant of $T_{1,7}^{10}$. Under this transformation we obtain in turn 4692, 483, 69, 69 and so 23 is a factor of 4692.

The prime 31 is an invariant of $T_{2,25}^{10}$. Starting with 18312 the repeated application of this transformation yields 3712, 792, 208, 240. Accordingly 18312 is not divisible by 31.

R. L. G.
M. R.

GLEANINGS FAR AND NEAR.

1529. As a professor of mathematics I am practically required by the ethics of the profession to be absent-minded, unmethodical, and inconsistent in many ways fatal to bibliographical excellence.—L. C. Karpinski. Preface to *Bibliography of Mathematical Works printed in America through 1850*. (1940.) [Per Prof. E. H. Neville.]

CAREERS FOR GRADUATES IN MATHEMATICS.

BY NANCY WALLS.

It is the intention of this paper broadly to survey present possibilities of work for honours graduates in mathematics. Detailed information, which obscures the picture and soon becomes out of date, will not be given.

The professional mathematician has a number of openings before him and a short account of these will first be given; I am here much indebted to Sir Edmund Whittaker who has kindly made available to me information concerning the careers of honours graduates of the Edinburgh school of mathematics. In dealing with conditions at different times one will normally be concerned only with a shift of emphasis; the main lines will be familiar. This must be noted, since the background against which prospects for mathematicians are at any time to be interpreted has radically altered during the last two decades.

In pointing out the possible careers before his graduates the head of a Department of Mathematics will usually give pride of place to the academic; not claiming it to be the most honourable, which would be unseemly, or the most lucrative, which would be untrue, but recognising that only in this vocation is one primarily concerned with Mathematics in itself. But primacy of place does not entail amplitude of description. The posts are small in number, they will usually be occupied by First Class men and those likely to be attracted to them will find their own stimuli. More necessary of mention are Lectureships in Technical Colleges and in Training Colleges, which are sometimes not considered by good students for whom they would provide a more grateful environment.

These last two possibilities connect naturally on the one hand with work in industry and on the other with school teaching. The latter profession is unlikely to be overlooked; the danger is rather that no other should be considered. Work in Training Colleges, in Education Departments and as Head Master are possible developments. Interesting variations are Naval Instructorships and teaching in the colonies.

A mathematician qualified also in physics may enter the Post Office Engineering Department of the Civil Service on the Executive level and there are positions in industry suitable for the same type of man, though it is customary to distinguish between the safe and steady employ of the Civil Service, with its familiar salary scales and pension scheme, and the less sure but possibly more repaying career in private industry. Of openings in industry and business not dependent upon physics that of the Actuary in Insurance is sufficiently specialised for individual mention. The man with a good honours degree in mathematics who has in addition those qualities making for success in the business world can gain a responsible and highly paid position if he is prepared after his degree to undertake the restrictive discipline of some years' steady office work combined with spare-time preparation for the successive parts of his actuarial examination.

For administrative positions in the Civil Service, with which we may group similar positions in the Colonial Civil Service and work in the Consular and Diplomatic Service, mathematics is not itself the primary object of study but is one of the possible main subjects to be offered by a candidate for the service, who will usually spend a year after taking his degree in exclusive preparation for his entrance examination. Mathematicians entering the Executive Class of the Civil Service may obtain appointments not only under the Post Office but in Inland Revenue and, more rarely, as cartographers. But Cartography leads us to Survey, Survey to Meteorology and so to the

whole gamut of posts for Scientific Officers. As it would be abortive to treat this point with no particular period in mind, we shall first consider some features of the present situation relevant to our discussion.

The interest roused during the war years by urgent appeals for "mathematicians" has been offset by the opinion of the *a priori* discerning and the *a posteriori* disillusioned that the real need, and that a transient one, was for young people capable of carrying out stock calculations and of operating precision instruments with reasonable speed and accuracy. This opinion was not completely unfounded: such a need certainly existed; but it masked a growing employment in more interesting and responsible work of those better qualified. The extension and development of Research Establishments under the auspices of different Government Departments is a direct result of the war. As partial and secondary effects important for our enquiry may be mentioned a tendency to closer cooperation between such establishments and the Universities, the abnormal demand for trained statisticians, the setting up of a Mathematical Section of the National Physical Laboratory at Teddington and the effect on both work in industry and research in Universities and Technical Colleges of the gradual release of information concerning research work carried out under conditions of secrecy during recent years.

Schemes of nationalisation are here important. The distinction made above between work in industry and in the Civil Service will still apply, but it seems probable that the line of demarcation will alter. Within the framework of an extending Civil Service the mutual stimulus of developments in different branches of mathematical technique may well increase. Industry, moreover, according to the recommendations of a recent Report, will be encouraged to work in closer cooperation with reorganised Technical Colleges.

Technical Colleges are not alone among educational establishments in being directly affected by present policy. The raising of the school-leaving age intensifies the demand for school teachers at a time when for some years the normal supply of mathematicians has been deflected towards physics and engineering. Prospective teachers must pass through Universities and Training Colleges also suffering from war-time short commons.

Such, briefly indicated, is the present situation. Leaving aside those careers in which work in mathematics as such is not the first concern, we see that it is one in which the need for mathematicians is still increasing. In teaching, so far as school work is concerned, demand seems likely to exceed supply for some years yet; the causes here are war-time dislocation and wastage and the present educational policy. In the higher branches of the profession the demand will again be great; the same causes will be effective and others also. Not only will the actual pedagogic work of University Departments be increased, but the repercussions of technical research reach even the purest strata of investigators. Already, during the war, we have become accustomed to the idea of a liaison officer between mathematical specialists in the Universities and research groups acting under Government Departments. Other forms of contact are also made. At University College, Southampton, we have for some time been giving lectures one day a week to the mathematical section of a near-by Government establishment. The members of this class are graduates; the best of them are extremely good mathematicians. Whilst the subjects of lecture courses are chosen as pertinently as possible for the work of the establishment, their treatment is definitely mathematical; application to particular problems is the business of the class. As is usual in a post-graduate course, there is constant appeal to a wider mathematical background than that provided by an undergraduate training, and the main courses are supplemented by shorter ones dealing with topics to which reference is repeatedly made. The assimilation

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of different themes into one body of thought has a twofold effect. The students begin to see the corpus of mathematical knowledge as something more than a useful armoury from which an occasional weapon may be selected, and the lecturer finds in the work of the applied scientist a disturbance that affects the rarest regions of his subject.

Work of this kind is still fairly new, but it provides growing demands on academic staff. If, as seems possible, the material for investigation in both pure and applied mathematics thrown up by technical research is submitted more fully to the Universities, the need for augmentation of staff will be yet greater.

As has already been implied, prospects in the Civil Service are extensive. Numerous branches, the Nautical Almanac Office, Survey at home and in the colonies, Statistical sections under various Ministries, Meteorological Research Stations, will claim a steady quota of scientific officers, whilst other establishments of paramount importance in the present state of world affairs are unlikely seriously to restrict their numbers.

Although some of the questions asked by war-time research workers have been of general and abstract nature—one meets applied scientists who profess themselves uninterested in actual solutions of certain differential equations but concerned rather with existence conditions—it is not only in this way that purely mathematical questions have been raised. The difficulties of dealing with observational data have often led to the growth of new branches of abstract science: some of the greatest names in the history of Pure Mathematics might be cited in this regard. When any particular problem is submitted to a mathematician the main concern of the propounder is usually with the answer; the interest for the mathematician lies in the method of solution and the question is one of central importance when known methods of approach cannot, with slight adjustment, be applied to it. For the transition from a science incapable of some particular application to a modified form is rarely brought about by mere annexation. The problem reverberates through the interlocking structure of mathematical thought.

The whole field of quantitative observation, from particular questions of interpretation of data to the fundamental problem of finding a satisfactory basis for the Theory of Statistical Mathematics, is to-day disturbed. Whilst omitting further discussion, one must emphasise the amount of work to be done in this field, for it has attractions both for those of abstract and philosophical inclination and for those concerned with the application to stubborn material of the most powerful methods of modern mathematics. Also, it offers excellent prospects of employment. The Universities cannot be uninfluenced by the demand for trained statisticians. That demand is such that in a few years time we may expect Lectureships in Statistics to be established where none have previously existed, whilst those schools of mathematics already provided with a statistical side will perhaps have been enabled so to extend these departments that they exhibit not only a flourishing school of post-graduate research but also a staff of research assistants at once learning the essentials of the practical side of their profession and relieving their senior colleagues of the routine work now increasingly provided by other departments.

The statistician who chooses to specialise in the application of his work to some particular science has many opportunities. In the Pure Sciences, in Biology and in Medicine, in Agriculture and Fishery, in Industrial Research and in many departments of the Civil Service or in the National Physical Laboratory the demands are increasing and the supply inadequate. The growth of this profession is more notable than that of any other here discussed.

The newly formed Mathematical Section at Teddington is concerned both

with statistics and with the application of the different techniques of numerical and mechanical methods of attack to the problems raised by other sections. It exhibits a keen interest in the development of these techniques and an appreciation of the vivifying effect on routine work of research into the more abstract problems raised.

The intensification of research into mathematical problems arising technologically much affects the prospects of mathematicians in industry. I well remember as a first-year undergraduate listening to a lecture from my Professor on the possibilities lying before us when we should at length have taken our degrees. In the course of this we were told of the further fortunes of the First Class Honours men of the Department over a number of years. About a third were Professors or Lecturers and a third were teaching; there were Actuaries and Civil Servants, "And one", said the Professor finally, raising his eyebrows, "one is doing research in the Cotton Industry". Only a few years ago it was rare for a mathematical graduate not qualified also in Physics to pass into industry. Already the increase is marked and a peak has certainly not been reached. The Government, engaged during the war in large-scale industry, has proved the worth not only of computers but also of trained mathematicians able both to give direction and advice to their juniors and colleagues and to retain cognizance of the latest advances in pure research. Industry as a whole will be affected as well by the principle of closer contact with pure science as by the results of research now beginning to be made public.

The chances of employment are good. The situation sketched above may seem encouraging for other reasons. Indeed, the stimulus given to work by appreciation and enthusiasm from other quarters is undeniable. But more sober thoughts supervene. One wonders whether Mathematics is to be mistress in her own house. A planned economy in which scholastic research is geared up to State needs may have much to offer the scientist as well in the excitement of new ideas as in well-paid employment, but the dangers are obvious. That apprentices to mathematics at this time should be aware of these dangers is vitally important. To broaden the picture, however, we conclude with some considerations not so far stressed.

At the close of the first world war the subject of Pure Mathematics was much subdivided. At the close of the second the confluence of different streams is already marked. Recent research has been characterized by an intense interest in underlying structure and by a degree of abstraction that might have caused head-shakings amongst those of more practical bent had theorems enunciated not proved as powerful of application as they are delicate of articulation. For the vision illuminating abstraction is integrity of thought; appreciation of structure precedes identification of the particular; *arsis* leads to *thesis*.

Nor is this confluence purely internal. The connections with experimental sciences are balanced by a more abstract integration. The growing attention given to the logical basis of the subject has strengthened the explicit connection with philosophy.

"So here I am, in the middle way, having had twenty years—
Twenty years largely wasted, the years of *l'entre deux guerres*—
Trying to learn to use words—
With shabby equipment always deteriorating
In the general mess of imprecision of feeling,"

says the poet; and in Poland, during those same years of *l'entre deux guerres*, the attempt to find a less inadequate tool than words for philosophical research brought together philosophers and mathematicians in the field of

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Mathematical Logic. The influence of the important schools of Warsaw, Lwów and Cracow, though now denuded and dispersed, is profound. A suggestion has recently been made that a course in logic such as that prescribed for pupils in the final secondary school year in Poland and Austria might with advantage be given here, or adapted for first-year students in Universities. There is a further possibility. We may see growing up under the wing of Mathematics schools not only of Statistics but also of Symbolic Logic. The central position of Mathematics between Applied Science and the Mental Disciplines gives both hope and opportunity to her disciples.

N. W.

1530. KING CHARLES' HEAD AND THE DEDEKIND SECTION.—It was without difficulty resolved that the indictment should be "for compassing the death of the king" . . . and that the decapitation should be laid only as the overt act to prove the compassing; but very puzzling questions arose whether the decapitation should be alleged to have taken place in the reign of Charles I or Charles II?—and against the peace of which Sovereign the offence should be alleged to have been committed.—Lord Campbell, *Lives of the Lord Chancellors* (1857), IV, p. 75. [Per Professor H. G. Forder.]

1531. Falmouth, October 5, 1837. . . . Sir Charles [Lemon] told us that Professor Airy . . . was so shy that he never looked a person in the face. A friend remarked to him, "Have you ever observed Miss —'s eyes? They have the principle of double refraction". "Dear me, that is very odd", said the philosopher, "I should like to see that; do you think I might call?" He did so, and at the end of the visit begged permission to call again to see her eyes in a better light. He, however, found it a problem which would take a lifetime to study, and he married her.—Caroline Fox, *Memories of Old Friends* (1883), p. 28. [Per Professor H. G. Forder.]

1532. September 9, 1848. When in Dublin, Sir William Hamilton mentioned to Airy some striking mathematical fact. He paused a moment. "No, it cannot be so", interposed Airy. Sir William mildly remarked, "I have been investigating it closely for the last few months, and cannot doubt its truth". "But", said Airy, "I've been at it for the last five minutes, and cannot see it at all".—Caroline Fox, *Memories of Old Friends* (1883), p. 286. [Per Professor H. G. Forder.]

1533. It is an odd thing to have had about £300 a year spent on your education until the age of seventeen and to be entirely unable to do long division, while regarding fractions and decimals as mysteries of the order of electrons. Because of these deficiencies I was unable to take [at an Agricultural College] the subject called "Survey". Book-keeping I could just manage—through having learned to credit the giver and debit the receiver, my addition still prevented me balancing a trial balance. As a light on the education of the females of the upper classes, I may here remark that neither Mary's nor Coney's knowledge of arithmetic was in any way superior to mine.—Frances Donaldson, *Approach to Farming* (Faber, 1941), pp. 86-7. [Per Mr. F. W. Kellaway.]

1534. And his chapter on "Poets' Numbers" is good reading, full of curious and entertaining instances. I could wish, however, that he had found room in it for the most remarkable numerical statement in English Literature, Gibbon's "A thousand swords were plunged at once into the bosom of the unfortunate Probus".—Edward Shanks, in *Sunday Times*, February 10, 1946; a Review of H. McKay, *The World of Numbers*.

THE SCIENTIFIC FILM ASSOCIATION.

A FIRST LIST OF FILMS ON MATHEMATICS.

This is an unchecked list compiled partly from personal knowledge of the films and partly from existing catalogues (which may be out of date). Members are invited to send in any corrections needed, together with two-line synopses of those films they have seen; comments on merit will also be appreciated.

A revised list will be issued as soon as sufficient information has been received in this way.

<i>Title</i>	<i>Country and date of origin</i>	<i>Size and type</i> (<i>Si.</i> = silent (<i>Sd.</i> = sound))	<i>Running time</i>	<i>Distributed by</i>	<i>Made by</i>	<i>Short synopsis</i>	<i>Remarks</i>
<i>The Theorem of Pythagoras</i>	BRITAIN, 1935	16 mm. Si.	4 minutes	R. A. Fairthorne, Kirk Michael, Hillfield Road, Farnborough, Hants. British Film Institute, 4 Gt. Russell St., W.C. 1	Fairthorne & Salt	Euclid's proof demonstrated by moving diagram	Teaching notes
<i>Euclid, 1. 32</i>	BRITAIN, 1935	16 mm. Si.	4 minutes	R. A. Fairthorne and B.F.I.	Fairthorne & Salt	Euclid's proof that the sum of the internal angles of a triangle is 180°	
<i>Equation</i> $\frac{x}{x} + x = 0$	BRITAIN, 1936	35 mm. Si. 16 mm. Si.	3 minutes	R. A. Fairthorne and B.F.I.	Fairthorne & Salt	A kinematic representation. Diagram based on the set-up of a differential analyser	Teaching notes

<i>Title</i>	<i>Country and date of origin</i>	<i>Size and type</i>	<i>Running time</i>	<i>Distributed by</i>	<i>Made by</i>	<i>Short synopsis</i>	<i>Remarks</i>
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Title	Country and date of origin	Size and type	Running time	Distributed by	Made by	Short synopsis	Remarks
<i>Equation</i> $\ddot{x} + x = A \sin nt$	BRITAIN, 1937	35 mm. Si. 16 mm. Si.	8 minutes	R. A. Fairthorne and B.F.I.	Fairthorne & Salt	A kinematic representation of the equation for simple harmonic motion, showing the effect of forced vibration	
<i>Generation of Involute Gear Teeth</i>	BRITAIN, 1939	35 mm. Si. 16 mm. Si.	4 minutes	R. A. Fairthorne and B.F.I.	Fairthorne & Salt	(1) The generation of involute gear teeth as the loci of points attached to a cross belt connecting two pulleys. (2) The generation of teeth as the envelope on the disc of a zig-zag cutter moving relatively to it. (3) The equivalence of the two methods	Teaching notes
<i>Hypocyclic Motion</i>	BRITAIN, 1938	35 mm. Si. 16 mm. Si.	9 minutes	R. A. Fairthorne and B.F.I.	Fairthorne & Salt	The motion is shown to be equivalent to a circle (diameter = ld) rolling inside another circle (diameter = $2d$)	Teaching notes

<i>Title</i>	<i>Country and date of origin</i>	<i>Size and type</i>	<i>Running time</i>	<i>Distributed by</i>	<i>Made by</i>	<i>Short synopsis</i>	<i>Remarks</i>
<i>Transfer of Power</i>	BRITAIN, 1938	35 mm. Sd. 16 mm. Sd.	22 minutes	Petroleum Film Bureau, 46 St. James Pl., London, S.W. 1	Shell Film Unit	The History of the toothed wheel, including the mathematics of the epicycloid and involute curves	
<i>Harmonic Motions</i> (1) Resultant Circle and Straight Line (2) Resultant Ellipses	BRITAIN	16 mm. Si., looped film or length of film	2 minutes	Dance-Kaufmann Ltd., 18 Upper Stanhope Street, Liverpool, 8	Dance-Kaufmann Ltd.	Illustration of the composition of simple harmonic motions	Notes
<i>Hypocycloid Gear, Ratio 2 : 1</i>	BRITAIN	16 mm. Si., looped film or length of film	2 minutes	Dance-Kaufmann Ltd.	Dance-Kaufmann Ltd.	The Cycle film shows a simple gear with the ratio 2 : 1	Notes
<i>Intermittent Movement, Ratio 8 : 1</i>	BRITAIN	16 mm. Si., looped film or length of film	2 minutes	Dance-Kaufmann Ltd.	Dance-Kaufmann Ltd.	Illustration of an intermittent driving-gear in which the average ratio between two shafts is 8 : 1	Notes

THE SCIENTIFIC FILM ASSOCIATION

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<i>Title</i>	<i>Country and date of origin</i>	<i>Size and type</i>	<i>Running time</i>	<i>Distributed by</i>	<i>Made by</i>	<i>Short synopsis</i>	<i>Remarks</i>
<i>Mouvements Vibratoires</i>	FRANCE, 1932 (?)	35 mm. Si. 16 mm. Si.	12 minutes	British Film Institute	Atlantic Films	Simple harmonic motion analysed by diagrams and models	
<i>Rate of Change</i>	BRITAIN, 1937	16 mm. Si.	9 minutes	EGS, Gebescope Film Library, Woodchester, Nr. Stroud, Glos.	Segaller-Smith	Exposition of the differential calculus. Demonstration of the basic idea of a limit by the motion of chords	
<i>Force of Gravity</i>	FRANCE, 1935	16 mm. Si.	15 minutes	EGS, Gebescope Film Library	Benoit-Levy	Comparison of rates of falling of different masses	
<i>Einstein's Theory of Relativity</i>	U.S.A., 1924	16 mm. Si.	30 minutes	Wallace Heaton Ltd., Film Library, 127 New Bond Street, London, W. 1	Fadman-Fleisher	Demonstration of the relativity of motion, direction, size, speed and time measurements	
<i>Frequency Curves</i>	U.S.A., 1929	16 mm. Si.	8 minutes	Wallace Heaton Ltd.	Eastman Kodak Co., Teaching Film Division	Explanation of the nature of a statistical frequency distribution and an interpretation of frequency curves	

E. L. HUPPERT,
Pure Science Committee.

December 1946.

MATHEMATICAL NOTES.

1949. *A circle connected with a triangle.*

The interesting properties of the circle through the points where the internal bisectors of the angles of a triangle meet the opposite sides, mentioned in Notes 1506 (February 1941) and 1799 (February 1945), are special cases of two general theorems, of which the first is known and the second believed to be new.

1. According to Note 1506, the considered circle passes through the Feuerbach point. But this point is the orthopole of the circumdiameter passing through the in-centre I , and the orthopole ω of any circumdiameter d is the centre of the rectangular hyperbola (H), circumscribed to the given triangle ABC , which is the counterpoint conic to d . If Q is any point on d , the counterpoint Q' of Q is such that, in the triangle $A'B'C'$, formed by the points where AQ' , BQ' , CQ' meet BC , CA , AB respectively, each vertex has the opposite side as polar with respect to (H), and it is well known * that, in that case, the circle $A'B'C'$ passes through the centre ω of (H). Hence the following theorem :

If Q' is the counterpoint of any point on the circumdiameter d of the triangle ABC , the circle through the points where AQ' , BQ' , CQ' meet the sides BC , CA , AB passes through the orthopole of d .†

The special case when Q is the in-centre I and ω the Feuerbach point, considered in Note 1506, has been mentioned by Meytelon.‡ The extension to exterior bisectors, considered in Note 1799, will be obtained when Q is one of the ex-centres I_a , I_b , I_c .

But the general theorem suggests also properties about other remarkable circles passing through the Feuerbach point.

For instance, the counterpoint of the Nagel point§ lies on the circumdiameter passing through I :

The Feuerbach point lies on the circle passing through the points where the sides touch the corresponding ex-circles.

Further, Gallatly || has proved that the Lemoine point of the triangle $I_a I_b I_c$ is the counterpoint of the centre of similitude of the triangle $I_a I_b I_c$ and the triangle formed by the contact points of the in-circle with the sides : this point is also on the circumdiameter passing through I :

The Feuerbach point is on the circle through the points where BC , CA , AB meet the lines joining A , B , C to the Lemoine point of the triangle $I_a I_b I_c$.

2. In Note 1799 it is shown that the circle through the points where the interior bisectors meet the opposite sides cuts off from the sides chords such that one of them is equal to the sum of the other two.

Let now P be any point having as barycentric coordinates α , β , γ , L , M , N the points where AP , BP , CP meet BC , CA , AB and X , Y , Z the points where the circle LMN cuts again BC , CA , AB respectively.

It will be convenient to take a circulation sense on the perimeter of the triangle ; the sides BC , CA , AB and the chords LX , MY , NZ will be denoted by a , b , c and x , y , z in value and sign.

* Salmon, *Conic Sections* (1879), p. 215, § 228, Ex. 5.

† See my paper on the Orthopole, *Tohoku Mathematical Journal*, 1926, p. 95.

‡ *Journal de Mathématiques élémentaires* (Vuibert), 1913-1914, p. 92. (Meytelon is a pseudonyme for Th. Lemoyne.)

§ Intersection of the joins of A , B , C to the points where the opposite sides touch the corresponding ex-circles.

|| *Modern Geometry of the Triangle*, p. 90.

Then, for instance,

$$\overline{BL} = a\gamma/(\beta + \gamma), \quad \overline{LC} = a\beta/(\beta + \gamma),$$

and relations such as

$$\overline{AM} \cdot \overline{AY} = \overline{AN} \cdot \overline{AZ}$$

may be written as follows :

$$- \overline{MA} (- \overline{MA} + \overline{MY}) = \overline{AN} (\overline{AN} + \overline{NZ}),$$

or

$$\frac{b\gamma}{\gamma + \alpha} \left\{ \frac{b\gamma}{\gamma + \alpha} - y \right\} = \frac{c\beta}{\alpha + \beta} \left\{ \frac{c\beta}{\alpha + \beta} + z \right\}.$$

Similarly,

$$\frac{c\alpha}{\alpha + \beta} \left\{ \frac{c\alpha}{\alpha + \beta} - z \right\} = \frac{a\gamma}{\beta + \gamma} \left\{ \frac{a\gamma}{\beta + \gamma} + x \right\},$$

$$\frac{a\beta}{\beta + \gamma} \left\{ \frac{a\beta}{\beta + \gamma} - x \right\} = \frac{b\alpha}{\gamma + \alpha} \left\{ \frac{b\alpha}{\gamma + \alpha} + y \right\}.$$

Multiplying these equations by α^2 , β^2 , γ^2 respectively and adding :

$$ax/\alpha + by/\beta + cz/\gamma = 0.$$

Hence the following theorem, in which segments are considered with their sign :

If ξ , η , ζ are the normal coordinates of a point P , the circle through the points where the joins from P to the vertices meet the opposite sides cuts off from the sides chords x , y , z such that

$$x/\xi + y/\eta + z/\zeta = 0.$$

When $\xi = \eta = \zeta$ or $-\xi = \eta = \zeta$, ... , we find the theorems given in Note 1799.

When $\xi = 1/a$, $\eta = 1/b$, $\zeta = 1/c$, P is the centroid, L , M , N are the mid-points of the sides and X , Y , Z the feet of altitudes ; then $ax + by + cz = 0$, which is very easy to check directly.

3. It is well known that AX , BY , CZ meet at a point P' . Let ξ' , η' , ζ' be the normal coordinates of P' ; then

$$x/\xi' + y/\eta' + z/\zeta' = 0.$$

If the circle through the points L , M , N , where AP , BP , CP meet BC , CA , AB cuts the sides again at X , Y , Z and if P' is the intersection of AX , BY , CZ , then, ξ , η , ζ and ξ' , η' , ζ' being the normal coordinates of P and P' , the chords LX , MY , NZ are in the same ratio as

$$\frac{1}{\eta\zeta'} - \frac{1}{\eta'\zeta}, \quad \frac{1}{\zeta\xi'} - \frac{1}{\zeta'\xi}, \quad \frac{1}{\xi\eta'} - \frac{1}{\xi'\eta}.$$

4. In connection with remarkable circles passing through the Feuerbach point, it may be remembered that this point is also on the pedal circle of the Nagel point.

This is a special case of the theorem according to which the pedal circle of any point on a circumdiameter d passes through the orthopole π of d .

A still more general theorem is that of Th. Lemoyne* : the pedal circle of any point on a straight line Δ with respect to a triangle has a constant power with respect to the orthopole S of that line.

The power is twice the product of the distance from S to Δ by the distance from Δ to the circumcentre.†

R. GOORMAGTIGH.

* *Nouvelles Annales de Mathématiques*, 1904, p. 400.

† Gallatly, *Modern Geometry of the Triangle*, p. 51.

1950. *On Note 1844.*

Prof. Hardy in Note 1844 proposes to check mathematically the popular assumption that in golf, steadiness as against brilliancy will tell more by strokes than by holes. He suggests pitting a mechanical player *A* who takes 4 strokes for every hole against a player *B* who has an equal chance x of making a supershot worth 2 strokes of *A*, and of making a subshot worth nothing. He finds that *B* is more likely to lose the holes.

But this assumption makes *B* a definitely inferior player, for on the green where *A* would hole in 1 stroke, *B* has no chance of a supershot, but risks a subshot. He will thus lose both by aggregate and by holes.

To make them equal, suppose instead that on the green, where *B* has no chance of a supershot, he does not risk a subshot, but holes in 1 like *A*.

Then the probabilities of *B* holing in respectively 2, 3, 4, 5 strokes are x^2 , $3x - 8x^2 + 6x^3$, $1 - 6x + 19x^2 - 28x^3 + 15x^4$, $3x - 18x^2 + 48x^3 - 60x^4 + 28x^5$.

His average number of strokes per hole is exactly 4. His chance of winning a hole is $3x - 7x^2 + 6x^3$, and his chance of losing $3x - 12x^2 + 22x^3 - 15x^4$. Chance of winning minus chance of losing is $5x^2 - 16x^3 + 15x^4$, which is definitely positive for all values of x .

The popular belief is thus completely justified. The conclusion is fairly obvious from a commonsense point of view, for *B* cannot win by more than 2 strokes, but he can lose by 3, 4, 5 or more. Hence if he wins as often as he loses, his aggregate of strokes will be dominated by the losses. Hence if his aggregate is the same he will win more holes.

D. E. LITTLEWOOD.

1951. *On Note 1862.*

Has Mr. Gibbins forgotten that, if we know one point common to two conics, we can write down the general equation of a conic through their other three points of intersection? No trickery is involved. If a conic S passes through P , its equation can be written as $S(x, y) - S(x_P, y_P) = 0$, and thus as $(x - x_P)u + (y - y_P)v = 0$ where u, v are linear functions of x and y ; in fact

$$u = a(x + x_P) + h'y + h''y_P + 2g,$$

$$v = h''x + h'x_P + b(y + y_P) + 2f,$$

where h', h'' are any two constants whose sum is $2h$. If, then, two conics S_1, S_2 both pass through P , their equations can be written as

$$(x - x_P)u_1 + (y - y_P)v_1 = 0, \quad (x - x_P)u_2 + (y - y_P)v_2 = 0,$$

and the coordinates of any point other than P which lies on them both satisfy the equation $S_3 \equiv u_1v_2 - v_1u_2 = 0$; in other words, S_3 is a conic through the other three points of intersection. If we leave one of the constants h_1', h_1'' , and one of the constants h_2', h_2'' arbitrary, we have already in S_3 the general conic through the three fixed points. Alternatively, if we give the constants particular values and obtain a particular conic, we have for the general conic the equation $\lambda S_1 + \mu S_2 + \nu S_3 = 0$; the circle through the three points is then

$$\begin{vmatrix} S_1 & a_1 - b_1 & h_1 \\ S_2 & a_2 - b_2 & h_2 \\ S_3 & a_3 - b_3 & h_3 \end{vmatrix} = 0.$$

We have supposed P to be an accessible point. If the common point is the point at infinity on the line $lx + my + n = 0$, the equations of S_1, S_2 can be expressed as

$$(lx + my + n)u_1 + v_1 = 0, \quad (lx + my + n)u_2 + v_2 = 0,$$

and S_3 follows as before. This is the modification required in the problem of normals to a parabola. If the parabola is

$$S_1 \equiv (lx + my)^2 + 2gx + 2fy + c = 0,$$

the Apollonian hyperbola through the feet of the normals from (α, β) is

$$S_2 \equiv (lx + my) \{m(x - \alpha) - l(y - \beta)\} + \{f(x - \alpha) - g(y - \beta)\} = 0,$$

and the accessible points common to the two curves lie on the conic

$$S_3 \equiv \{m(x - \alpha) - l(y - \beta)\}(2gx + 2fy + c) - (lx + my)\{f(x - \alpha) - g(y - \beta)\} = 0.$$

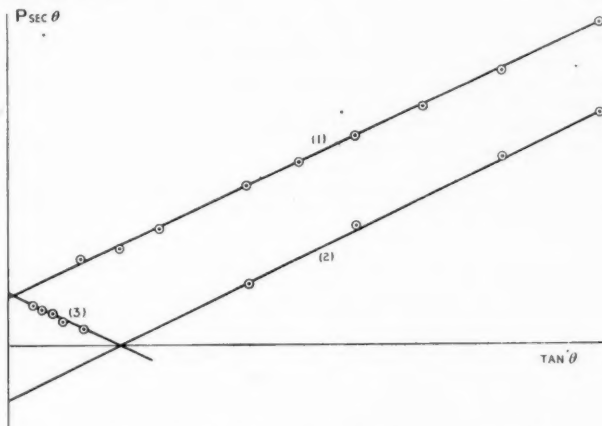
The co-normal circle of (α, β) is therefore

$$\begin{vmatrix} S_1 & l^2 - m^2 & 2lm \\ S_2 & 2lm & -l^2 + m^2 \\ S_3 & lf + mg & mf - lg \end{vmatrix} = 0.$$

The power of this circle for (α, β) is necessarily a multiple of $S_1(\alpha, \beta)$, since $S_2(\alpha, \beta)$ and $S_3(\alpha, \beta)$ are zero. E. H. N.

1952. *An experiment on motion on a rough inclined plane.*

Let W be the unknown weight of a sled in uniform motion upon a rough inclined plane of inclination θ . Let λ be the angle of friction. Let P_1 be the force parallel to a line of greatest slope of the plane required to maintain uniform motion up the plane. Let P_2 be the force of restraint in the same direction required for uniform motion down the plane when $\theta > \lambda$, and let P_3 be the force in the opposite direction required for uniform motion down the plane when $\theta < \lambda$.



The relationships between the forces can be expressed by the equations :

$$P_1 \sec \theta = W \tan \theta + W \tan \lambda,$$

$$P_2 \sec \theta = W \tan \theta - W \tan \lambda,$$

$$P_3 \sec \theta = -W \tan \theta + W \tan \lambda.$$

The variation of P_1 , P_2 and P_3 as θ varies can be represented in a simple manner if $P \times \sec \theta$ be plotted against $\tan \theta$. (1) and (2) are parallel lines with equal but opposite intercepts on the $(P \sec \theta)$ -axis, (1) and (3) have the

same intercept, and equal but opposite gradients, (2) and (3) have a common point on the $\tan \theta$ -axis where the sled moves uniformly down the plane and $\theta = \lambda$. A typical experimental graph is attached.

R. H. SMITH.

1953. *A single-sided doubly collapsible tessellation.*

After reading an interesting account of polyhedral linkages and collapsible surfaces by Michael Goldberg (*Nat. Maths. Mag.*, XVI, April 1942) the following construction occurred to me.

Cut out three equilateral triangles and three 60° rhombs all of unit edge. Arrange them alternately in a row to form a trapezium with base 5, top 4; and hinge them at the contiguous edges. Fold at the second and fourth hinges; tuck the last face under the first, and hinge them at the coincident edges. We thus get the Möbius zone shown in Fig. 1 (or its mirror-image form). It is a single-sided doubly collapsible surface. In each of its collapsed

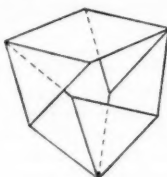


FIG. 1.

states it forms a regular hexagon: in one position a triangle coincides with one-half of a rhomb, and in the other position with the other half.

This construction can be repeated *ad infinitum* to form a tessellation, which is best begun in the following way: Make three zones as described above, all of the same kind (*R* or *L*). Take two and bring them together so that a rhomb of one coincides with a rhomb of the other and has a triangular plate over one half and another under the other half. Take the third zone and bring one of its rhombs into coincidence with a second rhomb of the first zone in the same way. Gum the coincident faces together. We now have a set of three connected zones. Make another such set; turn it bodily upside down and bring it up to the first set as shown in Fig. 2, where shaded areas indicate

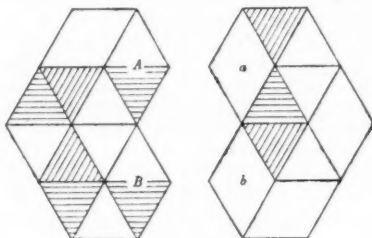


FIG. 2.

overlying triangles. Slide *a* under *A* and *b* under *B*, and when they coincide respectively, gum them together.

The resulting tessellation is, like the constituent zones, single-sided and doubly collapsible; and it contains fixed axes of rotation lying in its surface, namely the long diagonals of the rhombs.

SIDNEY MELMORE.

1954.

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1954. Note.

The following result may interest readers. There is at least one neat solution, and it is possible that the result can be extended.

Let $x_0, x_1, x_2, \dots, x_n$ be unequal numbers such that x_1, x_2, \dots, x_n are the roots of the equation

$$x^n + x_0 x^{n-1} + 2x_0^2 x^{n-2} + \dots + 2^{r-1} x_0^r x^{n-r} + \dots + 2^{n-1} x_0^n = 0.$$

Then the polynomial in t , of degree n , which takes the value -1 when $t = x_0$ and $+1$ when $t = x_1, x_2, \dots, x_n$, vanishes when $t = 0$. E. A. MAXWELL.

1955. On the operator D .

Mr. C. A. Coulson makes some sound points in his article "Some Difficulties in Teaching the D Method in Linear Differential Equations" in No. 287 of the *Mathematical Gazette*. The use of an operational method for obtaining the C.F., although not of course the best method in practice, has certain advantages in demonstrating both the significance of the C.F. and the scope of the operational processes.

I think, however, that the first method used for solving $dy/dx = ay$ (and a similar method used for a subsequent example) is dangerous. The student who, as Mr. Coulson admits, may question the validity of this may be even more embarrassing and ask what is wrong with the following:

$$\begin{aligned} y &= \frac{1}{D-a} \cdot 0 \\ &= \left(\frac{1}{D} + \frac{a}{D^2} + \frac{a^2}{D^3} + \dots \right) \cdot 0 \\ &= A + a(B + Cx) + a^2(E + Fx + Gx^2) + \dots \end{aligned}$$

The fact is, of course, that it is permissible in general to expand a function of D in ascending powers of D , but *not* in descending powers. This is in contrast with Heaviside's operational method in which expansion in descending powers of p is always valid, whereas expansions in ascending powers must be treated with caution.

It is hoped to make the validity of operational processes the subject of an article in a later issue of the *Gazette*. B. M. B.

1956. Exponential, logarithmic and circular functions.

Referring to Note 1805 (February, 1945), in which Mr. Tuckey advocates defining e as that value of a for which the gradient of $y = a^x$ at $x = 0$ is unity, it might be of interest to note that a similar method can be applied to the cosine and sine. Thus:

$$\frac{\cos(x+h) - \cos x}{h} = \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}$$

and

$$\frac{\sin(x+h) - \sin x}{h} = \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}.$$

The limit of $\frac{\cos h - 1}{h}$ is the slope of the graph of $\cos x$ at $x = 0$ and so is

visibly zero. The limit of $\frac{\sin h}{h}$ is the slope of $\sin x$ at $x = 0$ and is visibly some non-zero number depending on the scales (or unit) chosen, which will be unity if the proper choice is made for the unit of angle.

1958. *A peculiar reversion of numbers.*

The reversion of digits in

$$\overline{23} - 9 = \overline{32}, \quad \overline{65} - 9 = 56$$

leads to the following result :

if a and b are two groups each of k digits, then

$$ab - ba = (a - b) \cdot (9)_k,$$

where

$$(9)_k = 999 \dots \text{to } k \text{ digits.}$$

Examples.

$$(a) \quad 53 - 35 = 2 \cdot 9, \quad 68 - 86 = (-2) \cdot 9,$$

$$91 - 19 = 8 \cdot 9, \quad 09 - 90 = (-9) \cdot 9.$$

$$(b) \quad \overline{23} \overline{22} - \overline{22} \overline{23} = 99, \quad \overline{12} \overline{13} - \overline{13} \overline{12} = (-1) \cdot 99,$$

$$\overline{93} \overline{03} - \overline{03} \overline{93} = 90 \cdot 99, \quad \overline{02} \overline{84} - \overline{84} \overline{02} = (-82) \cdot 99.$$

$$(c) \quad \overline{342} \overline{152} - \overline{152} \overline{342} = 190 \cdot 999, \quad \overline{003} \overline{975} - \overline{975} \overline{003} = (-972) \cdot 999.$$

The proof of the result is immediate.

S. PARAMESWARAN.

1959. *Note on Pythagoras' Theorem.*

In textbooks and in books on the history of elementary mathematics, the opinion is very frequently expressed that Pythagoras did not prove the theorem which bears his name by the method given in *Euclid*, Book I, No. 47, but by a method which consisted of arranging portions of the figure in two different ways.

Usually the method by which this can be effected is given, but what is not given is any idea of any possible method by which Pythagoras might have arrived at such a proof, or the relationship of such a proof to the traditional figure of *Euclid*, I, 47.

The object of this note is to suggest a possible method by which Pythagoras might have been led to his result, and incidentally, the method does not require the use of the parallelogram or area theorems, but only involves congruent triangles.

In addition, it suggests a modified proof of *Euclid*, I, 47 (involving area theorems this time), which will be given separately at the end of the note.

Now let us endeavour to place ourselves in the position of Pythagoras with respect to the theorem.

We are told that the Egyptians knew that the 3, 4, 5 triangle was right-angled, and that they used its property ; while in the case of the isosceles right-angled triangle, any idler could verify the theorem for himself, in this case, by counting up the half or quarter squares in a tessellated pavement, and we presume that Pythagoras possessed this knowledge.

This might have led him on to the natural inquiry, that if the theorem were true for two types of right-angled triangles, would it be true generally?

We can imagine him drawing a right-angled triangle, ABC , with the right angle at A , and then describing the squares $ABFG$, $ACHK$ and $BCDE$ externally on the sides AB , AC , BC respectively (see figure).

It is then possible that he might have started to describe the square on the other side of BC and produced EB cutting FG at P . He would then probably notice the triangle FBP , and would have no difficulty in proving it congruent to ABC .

This might lead him on to produce DC to meet HK produced at Q , and again he would see that the triangle HQC was congruent to ABC .

The figure $PBCQ$ would therefore be a square, and in fact it is the square on the hypotenuse.

If then FG and HK were produced to meet at N , and FB and HC at O , the triangles NPQ and OCB are both easily seen to be congruent to ABC .

It is also clear that the rectangle $ABOC$ is twice the triangle ABC , while the rectangle $AGNK$ is *exactly* the same size as the rectangle $ABOC$.

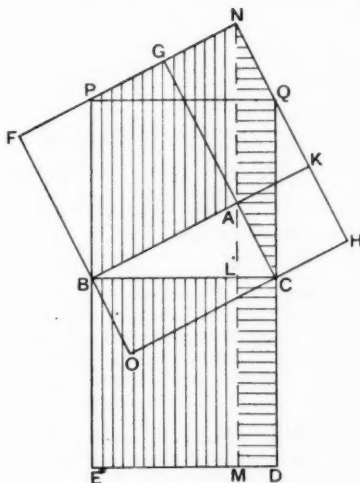
The proof is now clear. The figure $OHNF$ is equal to the square on BC plus four times the triangle ABC .

It is also equal to the square on AB , plus the square on AC , plus four times the triangle ABC .

Hence the square on BC is equal to the square on AB plus the square on AC , and the theorem is proved.

Incidentally, only the congruency theorems are used in the proof, and the theorem could be demonstrated to a class of juniors, far earlier than usual, if this proof is used.

The following very simple proof of Pythagoras' Theorem (which, however, requires area theorems) is suggested by the figure.



Since NP and AB are equal and parallel, it follows that NA is parallel to PB , and hence $NALM$ is perpendicular to BC .

Hence the square $ABFG$ = parallelogram $ABPN$

= rectangle $BLME$ (equal bases).

Also the square $ACHK$ = parallelogram $ACQN$

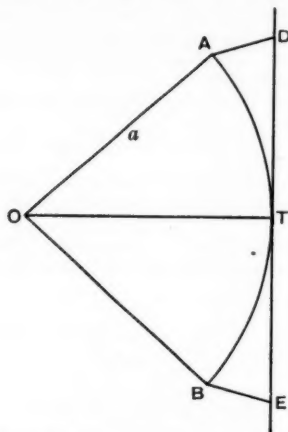
= rectangle $CLMD$.

Adding, the square on AB , plus the square on AC , is equal to the square on BC .

H. N. HASKELL.

1960. *A construction for the length of an arc of a circle.*

AB is an arc of a circle subtending an angle 2θ at the centre O ; DT is the tangent parallel to AB ; AD , BE make an angle $\theta/3$ with OT .



Then

$$\begin{aligned} DE &= 2a\{\sin \theta + (1 - \cos \theta) \tan \frac{1}{3}\theta\} \\ &\simeq 2a\{\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + (\frac{1}{3}\theta^2 - \frac{1}{24}\theta^4)(\frac{1}{3}\theta + \frac{1}{81}\theta^3)\} \\ &= 2a\{\theta + \theta^3(\frac{1}{120} - \frac{1}{72} + \frac{1}{162})\} \\ &= 2a\{\theta + \frac{1}{1620}\theta^5\} \\ &= \text{arc } AB, \end{aligned}$$

with percentage error $\theta^4/16 \cdot 2$, nearly.

Thus, if AB is a quadrant of a circle, the percentage error is approximately $1/50$. In this case, since the angle between OA and AD is 30° , the construction can be made with a set square, and even when the radius OA is 30 in. the difference between DE and the arc AB is less than 0.01 in. H. V. L.

1961. *On Note 1855.*

This method has considerable merit in special cases, for instance, when alternate powers of x are involved, but in general it does not yield such good results as the method I have recently employed.

Expanding the continued fraction

$$\frac{x}{a+} \frac{x}{b+} \frac{x}{c+} \dots$$

and the function required, in ascending powers of x and comparing coefficients, gives similar results when the expansion contains alternate powers of x , but in other cases better formulae are obtained. For example, by this method

$$e^x = (12 + 6x + x^2)/(12 - 6x + x^2)$$

with an error of $x^5/720$ in defect, from which we can write

$$a^x = (12\mu^2 + 6\mu x + x^2)/(12\mu^2 - 6\mu x + x^2)$$

with an error of $x^5/720\mu^5$ in defect, where $\mu = 1/\log_e a$.

Mr. C. G. Paradine's method gives

$$e^x = (12 + 9x + 5x^2)/(12 - 3x) - \frac{1}{3}x^2,$$

with an error of $x^5/480$ in excess.

As the errors have opposite signs, these formulae can be combined. Thus

$$e^x = \frac{1}{5} \left\{ \frac{3(12 + 6x + x^2)}{12 - 6x + x^2} + \frac{2(12 + 9x + 5x^2)}{12 - 3x} - \frac{x^2}{3} \right\},$$

where the error is about $x^6/2880$ in defect.

By comparing coefficients one step further, we can obtain still better results :

$$e^x = (120 + 60x + 12x^2 + x^3)/(120 - 60x + 12x^2 - x^3),$$

the error being $(x^7 + x^8)/100800$ in excess ;

and

$$(1+x)^n = \frac{12 + 6(2+n)x + (2+n)(1+n)x^2}{12 + 6(2-n)x + (2-n)(1-n)x^2},$$

with error $n(1-n^2)(4-n^2)x^3/720$. This is equivalent to Note 1875 (v), but is easier to handle.

The same method yields

$$\log_{10} (1+x) = \mu (30z - 8z^3)/(15 - 9z^2), \text{ where } z = x/(2+x).$$

This compares very favourably with the example given in Note 1879, § 3, the error being 3.5×10^{-16} in the example quoted.

R. H. BIRCH.

1962. On Newton's method.

In a specific problem we usually require the root of an equation to some definite degree of accuracy. Often, after much labour, we find that a particular approximation is no better than its predecessor. Hence it is well to recall the known but often neglected formula for the error, so that by using results already obtained we may easily estimate the order of the difference between any particular approximation and its successor, and hence decide whether it is necessary to find this successor.

Let a_1, a_2, a_3 be successive approximations, so that

$$a_2 - a_1 = -f(a_1)/f'(a_1), \quad a_3 - a_2 = -f(a_2)/f'(a_2).$$

Then

$$a_3 - a_2 = -f(a_1 + h)/f'(a_1 + h),$$

where

$$h = -f(a_1)/f'(a_1) \text{ and is small.}$$

That is,

$$\begin{aligned} a_3 - a_2 &= -\{f(a_1) + hf'(a_1) + \frac{1}{2}h^2f''(a_1) + \dots\}/\{f'(a_1) + hf''(a_1) + \dots\} \\ &= -h^2f''(a_1)/2f'(a_1), \end{aligned}$$

since h is small ; that is,

$$a_3 - a_2 = -\{f(a_1)\}^2 f''(a_1)/2\{f'(a_1)\}^3.$$

If this last quantity is of such order of smallness as not to affect the result to the degree of accuracy required, then a_2 is the required approximation and a_3 need not be calculated.

M. HUTTON.

1963. The Newton approximation.

1. The temptation is irresistible to add a third instalment to the story begun by Dr. Richmond in *Gazette*, XXVIII, p. 20, and continued by Mr. Lowry in *Gazette*, XXIX, p. 233.

We need to know the degree of accuracy to be expected from the application of Newton's method to

$$f(x) = x^{\frac{1}{2}(n+1)} - Rx^{-\frac{1}{2}(n-1)},$$

in which $f''(x) = (n^2 - 1)f(x)/4x^2$.

When $0 = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \frac{1}{6}f'''(a)h^3$,
it will be found that, for this function,

$$h = -\frac{f(a)}{f'(a)} - \frac{n^2 - 1}{12a^2} \left\{ \frac{f(a)}{f'(a)} \right\}^2 \dots\dots\dots(i)$$

E.g., if $n = 3$, $R = 2$;

$$f(x) = x^3 - 2x^{-1}, \quad f'(x) = 2x + 2x^{-2},$$

so that $f\left(\frac{5}{4}\right) = -\frac{3}{80}$, $f'\left(\frac{5}{4}\right) = 3.78$, $f\left(\frac{5}{4}\right)/f'\left(\frac{5}{4}\right) = -\frac{1}{100.8}$.

$$\text{Hence} \quad h = \frac{1}{100.8} + \frac{2}{3} \cdot \frac{16}{25} \cdot \left(\frac{1}{100.8}\right)^2,$$

the second term of which is less than 4.3×10^{-7} . Hence, correct to six places, we have

$$\sqrt[3]{2} = \frac{5}{4} + \frac{1}{100.8}.$$

By contrast, $a = 5/4$ applied to $x^3 - 2$ gives 1.26 with error -8×10^{-5} .

2. The contrast between the accuracies of $a - f(a)/f'(a)$ as calculated from $x^n - R$ and $f(x)$ as above is strongly brought out for $n = 2$, $R = 2$; the two values being $(a^2 + 2)/2a$ and $a(a^2 + 6)/(3a^2 + 2)$, respectively.

When $a = 3/2$ these are $17/12$ and $99/70$, or the second and fourth convergents after $3/2$ to $\sqrt{2}$ expressed as a continued fraction.

When $a = 17/12$, they are $577/408$ and $19601/13860$, or the fourth and eighth convergents after $17/12$. For this last result the correction is

$$\frac{1}{4} \cdot \frac{144}{289} \left(\frac{17}{6 \times 1055}\right)^2 < \frac{1}{8} \left(\frac{1}{6 \times 62}\right)^2 = \frac{1}{744^2} < 2.5 \times 10^{-9}.$$

Hence it gives $\sqrt{2}$ correct to eight places.

3. When $R = 1 + \epsilon$, $0 < \epsilon < 1$,

$$\begin{aligned} f(1) &= -\epsilon, \\ f'(1) &= \frac{1}{2}(n+1) + \frac{1}{2}(n-1)(1+\epsilon) = n\left\{1 + \frac{1}{2}(1-n^{-1})\epsilon\right\} \\ &= n\kappa, \text{ say.} \end{aligned}$$

Hence $(1+\epsilon)^{1/n} = 1 + \frac{\epsilon}{n\kappa} + \frac{n^2-1}{12} \left(\frac{\epsilon}{n\kappa}\right)^2$, by (i),

$$\text{or} \quad n\{(1+\epsilon)^{1/n} - 1\} = \frac{\epsilon}{\kappa} + \frac{1}{12} \left(1 - \frac{1}{n^2}\right) \left(\frac{\epsilon}{\kappa}\right)^2.$$

If $n \rightarrow \infty$, $\kappa \rightarrow 1 + \frac{1}{2}\epsilon$,

$$\text{and} \quad \log_e(1+\epsilon) = \frac{\epsilon}{1 + \frac{1}{2}\epsilon} + \frac{1}{12} \left(\frac{\epsilon}{1 + \frac{1}{2}\epsilon}\right)^2;$$

or, putting $\epsilon/(1 + \frac{1}{2}\epsilon) = \eta$,

$$\log \{(1 + \frac{1}{2}\eta)/(1 - \frac{1}{2}\eta)\} = \eta + \frac{1}{12}\eta^2.$$

E.g., $\log_e 1.25 = 2/9$, $\log_e 1.28 = 14/57$, both errors being about 10^{-3} . These give $\log 2 = .69$, $\log 10 = 2.3$.

The same method applied to $x^n - (1 + \epsilon)$ gives only $\log_e (1 + \epsilon) = \epsilon$ with error $-\frac{1}{2}\epsilon^2$.
N. M. GIBBINS.

1964. *Bernoulli's and Euler's numbers.*

If we write

$$y = \sec x + \tan x = S_0 + S_1 \frac{x}{1!} + S_2 \frac{x^2}{2!} + S_3 \frac{x^3}{3!} + \dots,$$

the odd coefficients, S_{2n-1} , are "prepared Bernoullians",

$$S_{2n-1} = 2^{2n}(2^{2n} - 1)B_{2n-1}/2n,$$

and the even coefficients S_{2n} are the Euler numbers.

A recurrence formula for these is often obtained by a somewhat laborious process of picking out coefficients, but the problem can be treated quite straightforwardly.

We have

$$w = \sec x = S_0 + S_2 \frac{x^2}{2!} + S_4 \frac{x^4}{4!} + \dots,$$

and $y' = \sec x \tan x + \sec^2 x = wy$.

Differentiating n times by Leibnitz' theorem,

$$y_{n+1} = wy_n + \binom{n}{1} w_1 y_{n-1} + \binom{n}{2} w_2 y_{n-2} + \dots + w_n y.$$

Now put $x = 0$; the values of w_{2r} , w_{2r+1} , y_p for $x = 0$ are S_{2r} , 0, S_p . Hence

$$S_{n+1} = S_0 S_n + \binom{n}{2} S_2 S_{n-2} + \binom{n}{4} S_4 S_{n-4} + \dots,$$

where $S_0 = y(0) = 1$.

J. GORDON SKELLAM.

1965. *An identity in continued fractions.*

If $S_N \equiv S_N(x_0, y_1, x_1, y_2, x_2, \dots, y_N, x_N)$

$$\equiv x_0 + \frac{1}{y_1 + \frac{1}{x_1 + \frac{1}{y_2 + \dots \frac{1}{y_N + \frac{1}{x_N}}}}}$$

then

$$S_N \equiv \sum_{n=0}^N \left\{ x_n \sqrt{\left(\frac{\partial S_N}{\partial x_n} \right)} \right\}.$$

Proof. Let

$$R_n = \frac{1}{y_{n+1} + \frac{1}{x_{n+1} + \dots \frac{1}{x_N} = \frac{p_n}{q_n}}, \dots \dots \dots (1)$$

where p_n/q_n is the "reduced" form; then

$$S_N = x_0 + \frac{1}{y_1 + \frac{1}{x_1 + \dots \frac{1}{x_n + R_n} = \frac{P_n}{Q_n}}, \dots \dots \dots (2)$$

and

$$S_N + (\Delta S_N)_n = x_0 + \frac{1}{y_1 + \frac{1}{x_1 + \dots \frac{1}{x_n + R_n + \lambda} = \frac{P_n'}{Q_n'}}, \dots \dots \dots (3)$$

$$\text{By (2) and (3), } (\Delta S_N)_n = \frac{P_n' Q_n - P_n Q_n'}{Q_n' Q_n} = \frac{1}{Q_n' Q_n}; \dots \dots \dots (4)$$

the numerator is positive because only odd terms (the x_n terms) are considered, and unity by the fundamental property of continued fractions.

Now each term occurs linearly in the numerator and denominator, and hence, regarding $x_n + R_n$ as a term, we can write

$$\left. \begin{aligned} P_n &= A_n(x_n + R_n) + B_n, \\ Q_n &= C_n(x_n + R_n) + D_n. \end{aligned} \right\} \dots\dots\dots(5)$$

Also

$$\frac{P_n'}{Q_n'} = \frac{A_n(x_n + R_n + 1/\lambda) + B_n}{C_n(x_n + R_n + 1/\lambda) + D_n},$$

which on reduction and comparison with (5) gives

$$Q_n' = \lambda Q_n + C_n. \dots\dots\dots(6)$$

Writing $1/\lambda = \Delta x_n$, and letting $\Delta x_n \rightarrow 0$,

$$Q_n' \Delta x_n \rightarrow Q_n. \dots\dots\dots(7)$$

and hence, after substituting in (4) and taking the square root,

$$\sqrt{\left(\frac{\partial S_N}{\partial x_n}\right)} = \frac{1}{Q_n}; \dots\dots\dots(8)$$

hence

$$\begin{aligned} \sum_{n=0}^N \left\{ x_n \sqrt{\left(\frac{\partial S_N}{\partial x_n}\right)} \right\} &= \sum_{n=0}^N \left(\frac{x_n}{Q_n} \right) \\ &= \sum_{n=0}^N \left\{ \frac{x_n}{C_n(x_n + R_n) + D_n} \right\}, \text{ from (5)} \\ &= \sum_{n=0}^N \left\{ \frac{x_n}{C_n(x_n + p_n/q_n) + D_n} \right\}, \text{ from (1)} \\ &= \sum_{n=0}^N \left(\frac{x_n q_n}{Q_N} \right), \text{ on reduction. } \dots\dots\dots(9) \end{aligned}$$

Now

$$\begin{aligned} \frac{p_{n-1}}{q_{n-1}} &= \frac{1}{y_n + x_n + q_n} \frac{p_n}{q_n}, \text{ from (1)} \\ &= \frac{p_n + x_n q_n}{q_n(1 + x_n y_n) + p_n y_n}, \end{aligned}$$

which is now reduced, and so, comparing the numerators,

$$p_{n-1} = p_n + x_n q_n. \dots\dots\dots(10)$$

By (9) and (10),

$$\begin{aligned} \sum_{n=0}^N \left(\frac{p_{n-1} - p_n}{Q_N} \right) &= \frac{p_{-1} - p_N}{Q_N} \\ &= P_N / Q_N = S_N. \end{aligned}$$

Hence the identity is proved. It can also be proved by induction.

There is a dual identity easily deducible from this: with $x_0 = 0$ and x_N infinite,

$$\sqrt{(-1)} + \sum_{n=1}^N \left\{ y_n \sqrt{\left(\frac{\partial S_N}{\partial y_n}\right)} \right\} = 0.$$

These identities were discovered and used in the investigation of properties of a particular electric circuit. E. H. MANSFIELD.

1535. October 7, 1847. Professor Adams speaks of those about whom the English scientific world is so indignant, in a spirit of Christian philosophy, exactly in keeping with the mind of a man who had discovered a planet.—Caroline Fox, *Memories of Old Friends* (1883), p. 278. [Per Professor H. G. Forder.]

REVIEWS.

Space and Spirit. By Sir EDMUND WHITTAKER. Pp. 149. 6s. 1947. (Nelson)

It is no accident that mathematical physicists have at all times been interested in the philosophic problems arising out of our knowledge of the physical universe, for they, of all scientists, are most concerned with the comprehension of the whole by the human mind. In this small book, an eminent mathematician returns to the old problem of the bearing of natural science upon belief in God. Tracing the profound changes which have taken place in the conception of the universe since the time of St. Thomas Aquinas, Sir Edmund Whittaker sketches the alternation of Platonic and Aristotelian outlook through the ages. His aim has been to indicate—for the consideration of theologians who are also men of science—that the obstacles to natural theology are less formidable than has been supposed, and moreover that scientific discovery and understanding has opened up new prospects and possibilities to the advocate of belief in God.

The book is the substance of Donellan lectures delivered in Trinity College, Dublin, in 1946. Like all attempts to present in simple language the possible consequences of theories which are essentially mathematical in nature, the mathematical reviewer cannot help but wonder how far the non-mathematical reader will be able to grasp the argument. Nevertheless the book is very readable and will stimulate many to a desire to understand more deeply, especially as it is clear that twentieth-century physics has called a halt to the mechanistic determinism of earlier days. For those who have some acquaintance with relativity and quantum theory, Sir Edmund has written a book that will be much appreciated. E. C.

David Rittenhouse. By EDWARD FORD. Pp. vi, 226. 15s. 6d. 1946. (University of Pennsylvania Press; Geoffrey Cumberlege, Oxford University Press)

The subject of this biography enjoyed a considerable reputation as a scientist prior to, during, and after the American War of Independence. Born in 1732 on a farm a score of miles from Philadelphia, he was self-taught in mathematics and astronomy; he became an expert clock-maker, and it appears that some of his clocks are still performing their functions with complete satisfaction. An elaborate orrery which he constructed early in his career became an object of admiration in the State of Pennsylvania, and his reputation was enhanced to such an extent that he was appointed on several occasions to survey the State boundaries. The transit of Venus in 1769 gave Rittenhouse an opportunity of applying his skill as an instrument-maker and of demonstrating his intellectual gifts as an astronomer; his own observations were made under perfect conditions, and his calculation of the Solar Parallax—obtained by combining his own observations with those made elsewhere—gave a result which placed the Sun at a mean distance from the Earth of just less than 93 million miles, not far from the present-day value; undoubtedly this was a lucky result, for the timing of each of the planet's contacts with the Sun's limb was uncertain by several seconds owing to the effect of the planet's atmosphere, an explanation that escaped Rittenhouse although well understood by European observers of the transit.

In Philadelphia Rittenhouse became involved in public affairs, becoming State Treasurer during the Revolution period and holding various other posts in succession. Despite his official duties his scientific interests were not allowed to languish, and these covered a wide field—meteors, observations of eclipses, experiments in magnetism, the discovery of the principle of the

diffracting grating, the determination of planetary orbits, the discovery of a comet, and so on.

Rittenhouse's scientific eminence was recognised by his election to the Presidency of the American Philosophical Society and to the Royal Society in 1795, the year before his death.

The biography is pleasantly written, and gives an interesting picture of scientific and official life in a British Colony soon to become one of the most important States in the Union.

W. M. S.

A Locus with 25920 Linear Self-transformations. By H. F. BAKER. Pp. xi, 107. 8s. 6d. 1946. Cambridge Tracts in Mathematics, No. 39. (Cambridge University Press)

Professor Baker has long been the father of the current school of English geometers, though the growing number of the pupils of his own pupils is beginning to qualify him for the office of grandfather. All his followers among the first and second generations of his students, as well as his many friends and colleagues abroad, particularly in Italy, will be delighted by this little book, and its impressive evidence of his undiminished skill.

The development of a great analyst into the founder of a famous geometrical school must have puzzled many of Professor Baker's contemporaries. Yet this tract is further evidence, if it were needed, that there was no unnatural conversion. It has long been known (and is demonstrated in Jordan's classical *Traité des Substitutions* of 1870) that the group of the trisection of the periods of a theta function of two variables is tied up with the group of the substitutions on the twenty-seven lines of a cubic surface. This group, of order 51,840, has a simple subgroup of index two which can be represented by linear transformations in a space of four dimensions which leave invariant a quartic primal of this space. In this book, which barely mentions the function-theoretic aspects, the geometrical properties of this *Burkhardt primal* are developed, together with the theory of various associated configurations. Among these latter may be mentioned the forty-five nodes of the primal (whose symmetry is shown to advantage in Professor Baker's notation), which correspond to the forty-five tritangent planes of a cubic surface in ordinary space. Each node is a vertex of three of a set of twenty-seven simplices of the four-dimensional space, each of whose five vertices is a node of the primal, these simplices corresponding, as one would expect, to the lines of the cubic surface. The primal can be projected into itself from each of the forty-five nodes by a harmonic involution of the four-fold space: these forty-five projections generate the group of linear transformations which can, however, be derived from a far smaller number of its elements. The geometrical significance of the subgroups leaving invariant some configurations associated with the primal is brought out in a masterly manner.

It is unlikely that this tract will be of interest to any but professional geometers, and perhaps a few algebraists. But the pleasure they will derive from reading it should more than justify the labour that the author has expended on it. Although the book is only a small one, there is a great deal of matter in it, and the student who is prepared to give to it the attention it deserves can reap a rich harvest from almost every page.

D. B. S.

Analytische Geometrie der Ebene und des Raumes. By RUDOLF FUETER. Pp. 180. 22.50 Swiss fr. 1945. (Verlag Birkhäuser, Basel)

This book is based on a course of lectures given for many years to first-year students in the University of Zürich. The emphasis is on the underlying ideas rather than on formal technique, and the fundamental ideas of the analytical geometry of the plane and of three-dimensional space are discussed in an

admirable fashion. The book goes as far as quadrics, via the straight line, the plane and the sphere. Paper and print are excellent, and the book does not conform to any economy standard. I would especially recommend it to anyone beginning to read German mathematics, for here is a book he will be able to follow very easily, and in doing so he will both enjoy himself and learn all the usual mathematical terms. D. PEDOE.

Vers l'infiniment petit. Par ARMAND DE GRAMONT. Pp. 247. 160 fr. 1945. L'avenir de la Science, 22. (Gallimard)

The book is principally an account of the development of the microscope and its applications. It contains a vast amount of extremely well-digested and attractively presented information. The scientific principles both of the techniques and their applications are expounded with the aid of numerous well-executed line-diagrams, and some of the main achievements are illustrated in nine excellent plates. There is a chapter on the "electron microscope", and the last chapter gives a brief account of "radioactive indicators" for tracing the behaviour of chemical elements in living organisms. (The book was, however, apparently written before the very recent developments of "phase-contrast" methods in microscopy.)

Such a success in the way of popular exposition, in the best sense, would deserve a much longer notice in a more appropriate journal. But it is not in any way mathematical, and so this is scarcely the place for further detail.

W. H. McC.

The Common Sense of the Exact Sciences. By W. K. CLIFFORD. Edited with a preface by KARL PEARSON. Newly edited, with an introduction, by JAMES R. NEWMAN. Preface by BERTRAND RUSSELL. Pp. lxvi, 249. 15s. 1947. (Sigma Books, London)

No doubt many of us at some time or other have come on this book of Clifford's and, whatever our mathematical level at the time, must have been astonished and delighted by the superb clarity and easy mastery of that long out-of-print volume. Mathematics has changed and grown since 1885, but so often on lines which Clifford foresaw that sixty years have not unduly dimmed the freshness and value of his writings. Like Riemann, whom he so much admired, Clifford was both versatile and original in thought; like Riemann, too, disease brought a brilliant life to an untimely end. His life has been recounted in a memoir by Sir Frederick Pollock, prefixed to Clifford's *Lectures and Essays*, and in it there is one passage, too well known and too lengthy to quote in full, from which a sentence or two may be extracted, describing an ideal which all teachers must admire, though most of us will do so despairingly. Pollock found trouble in grasping Ivory's theorem, but had his difficulties cleared away while talking with Clifford: "he appeared not to be working out a question, but simply telling what he saw . . . real and evident facts which only required to be seen . . . the only strange thing was that anybody should fail to see it in the same way."

In the present book there are five chapters: Number; Space; Quantity; Position; Motion. The manuscript of the book was incomplete when Clifford died. He had seen proofs of Chapters I and II, but Karl Pearson added half of III, all of IV, and completely revised V, working on the ideas which Clifford had planted in Pearson's agile and receptive mind. It is thus difficult to say what is Clifford and what is Pearson expounding Clifford's ideas. But the power of vision is evident, as it is in Clifford's *Elements of Dynamics*. The writer is not grinding perfect syllogisms out of a logical mill, he is—whether he be Clifford or Clifford speaking through Karl Pearson—merely describing what he can see. This clarity of sight is most valuable in the geometrical

and physical sections, particularly those which foreshadow present-day ideas about space-curvature, where prophetic foresight has been superbly vindicated. The technique in each section is the same: a start is made from the simplest concrete things, the number of letters in a word, the bounding surface of a solid object, the answer to the question "How do I get from here to the George and Dragon?", and in a few pages we are grappling with real number, topology, vectors, and grappling with them in a spirit of confidence induced by the writer's wonderful power of showing that apparently hard things are really easy.

Clifford had the defects of his qualities, and it is possible that his very clearness of vision has its drawbacks, that the startling rapidity of exposition sometimes leaves no lasting illumination. He saw so much so clearly that he could hardly believe in the existence of anything he could not see. In his essays, there is a hard certainty about his certainties and about his doubts which leaves an impression of *naïveté*, natural enough when we remember what, in view of the amount and quality of his work, is so easy to forget, that he died at thirty-four. Perhaps to get a balanced view of the man as he was, Pollock's memoir, on which the editor of this new edition has mainly relied for his own introduction, should be supplemented by a glance at W. H. Mallock's half-forgotten little masterpiece, *The New Republic*. The slight but vivid portrait of "Mr. Saunders" may be mildly malicious, but it was surely drawn from life. There is obvious distortion in accepting as Clifford's such assertions as: "progress is such improvement as can be verified by statistics", "the generation that travels 60 miles per hour is at least five times as civilised as the generation that travels only twelve", yet it is only a distortion, not a falsification, of Clifford's fervent belief in the regenerative power of rational and scientific thought.

The Association's list of books suitable for school libraries recommends this volume, and we are grateful to the publishers for making it once more available. There is a new preface by Bertrand Russell, from which we must quote: "When I was fifteen years of age I read it . . . with passionate interest and with an intoxicating delight in intellectual clarification. . . . Now . . . I find that it deserved all the adolescent enthusiasm that I bestowed upon it when I first read it."

T. A. A. B.

Cours de mécanique rationnelle. I. Dynamique du point matériel. Par J. CHAZY. 3rd edition. Pp. 482. 825 fr. 1947. (Gauthier-Villars)

The French language seems particularly well suited to the exposition of rational mechanics; Professor Chazy's book does not, of course, rival the majestic sweep of Appell's classic work, nor has it the concise clarity of de la Vallée Poussin's slim volumes, but it is lucid and penetrating, and on several topics provides discussions not readily accessible elsewhere.

The present third edition of the section on particle dynamics contains eight chapters. Chapter I: Vector Theory, is a pleasant account of the elements, but in the later chapters a more whole-hearted use of vector methods would have been acceptable. Chapter II: Principles of Mechanics, is the crux of the whole book; dynamics, however academically treated, rests ultimately on physical bases, and the translation from physical phenomena to a set of postulates on which a purely logical structure can be raised is a task which an honest author must tackle. M. Chazy's account is on the whole reasonable and satisfactory. Chapter III: General Theorems on the motion of a particle, does not make the sequence of theorems appear as parts of a connected whole, though separate sections are individually clear. There is an excellent discussion of a matter of some academic interest, often passed over in textbooks, the possibility of the introduction of extraneous solutions into a dynamical

problem by taking first integrals, such as the energy equation, as equivalent to the equations of motion. Chapter IV: rectilinear motion of a particle, pays particular attention to complete discussions of the possible solutions. Chapter V: curvilinear motion, deals with the parabolic and the resisted trajectory, motion under central forces, and has a section on the motion of an electrified particle in a magnetic field. The discussion of the character of the resisted path of a projectile is particularly good, demonstrating how much can be said without explicit integration, and making effective use of the hodograph. Chapter VI: motion of a particle on a curve, deals with the smooth curve, the rough curve—this case very lucidly treated—, and the moving curve. Chapter VII: motion of a particle on a surface, has a thorough analysis of the spherical pendulum, among other matters. Chapter VIII: motion related to the earth, has the rather misleading “force centrifuge composé”, but is helpful on Foucault's pendulum; I know of no account which is unshakably convincing.

The subject-matter is the author's selection and we can not deplore that he prefers to write clearly and fully on his chosen topics, however much we may regret certain omissions. The absence of a specific section on Kinematics is not serious, but has caused some inconsistency: thus the polar components of velocity and acceleration are derived in the vector chapter, but not the intrinsic components, though these are used later in the book. There is a most interesting 70-page appendix of problems proposed for the “Certificat de Mécanique Rationnelle” of the Faculty of Sciences at Paris. This opportunity of studying the actual scope and nature of the examination is most welcome, but in view of the title of this volume, we are surprised to find that most of the questions deal with rigid dynamics.

Granted that the flavour is academic, the volume is an excellent example of French lucidity of exposition.

T. A. A. B.

Iohannis Henrici Lamberti. Opera Mathematica. I. Commentationes Arithmeticae, Algebraicae et Analyticae, 1. Edited by A. SPEISER. Pp. xxxi, 558. 1946. (Orell Füssli Verlag, Zürich).

A man of outstanding genius is likely not only to give his name to an age but also to cause stars of lesser magnitude to be forgotten. We naturally think of mathematics in the middle of the eighteenth century as that of the Eulerian age, but many of Euler's contemporaries were men of genius only just less than his. Of these, Lambert is surely one, though his name is now half-forgotten. He touched the mathematics of his time at many points, and the number of problems to whose solution he made significant contributions is not small: we may instance, among others, the theory of numbers, continued fractions, hyperbolic functions, elliptic integrals, non-euclidean geometry, perspective, logic, calculating machines, insurance.

Lambert's life (1728–1777) was not entirely fortunate; he died at 49, while his earlier years were oppressed by a harsh poverty, which drove him to ingenious shifts merely to procure candle-light for his studies. Such circumstances are not rare in the history of science, and we may be tempted to speculate on what the result of an easier upbringing might have been. If we are naturally inclined to suppose that mathematics might have profited, we should on the other hand be willing to admit that Toynbee's doctrine of “Challenge-and-Response” may apply here, a challenge of the maximum bearable intensity producing a maximum response. This has its application to teaching. There are two objections to “Mathematics made easy”; one is that such expositions are usually much more obscure than the orthodox textbooks, the other is that by depriving mathematics of its challenge we may well fail to evoke the pupil's response. Is this the underlying philosophy

of the old dictum: "It does not matter what you teach as long as the boys don't like it"?

Of Lambert's achievements, that one which is most likely to preserve his name longest from oblivion is his proof that π is irrational. The present volume does not contain the final memoir on this topic, which will appear in the forthcoming second part. But the present volume contains a great deal of interesting matter; the series

$$x = \frac{q}{p} - \frac{q^m}{p^{m+1}} + \frac{mq^{2m-1}}{p^{2m+1}} - \frac{m(3m-1)}{2} \frac{q^{3m-2}}{p^{3m+1}} + \frac{m(4m-1)(4m-2)}{2 \cdot 3} \frac{q^{4m-3}}{p^{4m+1}} - \dots$$

for a root of $x^m + px = q$, an obvious forerunner of later important development processes; a series for the arc-length of an ellipse; tables and manipulations which show Lambert's interest in number problems and in continued fractions; and an interesting memoir on rectification and quadrature by inscribed and circumscribed polygons.

The book has been most pleasingly produced and is clearly and elegantly printed. It contains a curiously fascinating portrait and a reprint of the *Éloge* on Lambert presented by J. H. S. Formey to the Royal Academy of Berlin. Generous patronage has made the fine work of the publisher and of the editor possible; Professor Speiser, as editor, has contributed a twenty-page introduction commenting concisely and pointedly on the papers included in this part. The second part of the first volume may be expected to follow this one very shortly.

Students of the growth and development of mathematics will find much to appeal to them in this collection: the leisurely pace, the copiousness of exposition, make it possible sometimes to see the new and unexpected ideas take form in the author's mind, while sometimes we may wonder why so acute an observer has missed what to us seems so obvious. Only those to whom the work of the "Old Masters" makes no aesthetic appeal should leave this volume unopened.

T. A. A. B.

Statistical Quality Control. By E. L. GRANT. Pp. xii, 563. 25s. 1946. Industrial Organisation and Management Series. (McGraw-Hill)

Statistics as a discipline in this country began to develop away from economics and the theory of errors of astronomical observation when Karl Pearson expressed mathematically some ideas, such as correlation, of Sir Francis Galton, and thus and in other ways opened up new fields. Of his school at University College, London, some of the most famous British and American statisticians are pupils. In 1932 the American debt was in part repaid by a visit to this country of Dr. Walter A. Shewart, of the Research Laboratories of the Bell Telephone Company, New York. This visit aroused considerable interest, and in 1933 the Royal Statistical Society set up an Industrial and Agricultural Research Section. The main new weapons used by the section were two, both rather in the realm of small sampling theory than of the classical large sample theory, and both made little use of the idea of correlation: they were the Analysis of Variance, pioneered by R. A. Fisher, and what is known in England as Q.C. Some of the theoretical foundations of the last were and are still being developed in Karl Pearson's laboratory by his son, Egon S. Pearson, now Professor in his father's place. During the war, as technical readers of the *Production and Engineering Bulletin* of the Ministry of Labour and National Service will have become aware, there were being made considerable extensions in connection with Quality Control. The Ministry of Production issued a leaflet on the topic, and the Ministry of Supply set up a special branch under the Controller of Physical Research and

Signals Development. The head of the Advisory Service of the Ministry of Supply was a well-known statistician and British experience was closely linked with that in the U.S.A., growing under the Office of Production Research and Development of the War Production Board.

The author of this book is Professor of Economics of Engineering at Stanford University and is, it appears, author of an earlier book, *Principles of Engineering Economics*. His present book is developed, he says, from courses that he has given in the past seventeen years, and, in particular, from intensive courses given during the war. A number of his examples are based on this wartime experience, especially from the West (Pacific) Coast, but some illustrative examples from British practice and publications are given, while there are references to the experience of the Australian Ministry of Munitions. The differences in practice and in symbolism between these various countries are pointed out.

Quality Control, or, as the author prefers to call it, Statistical Quality Control, is a statistical device for determining how uniform are items of mass-production. In cases of this type of work, we are aiming at an article with certain specified qualities: each of these qualities can be measured and the article can be made to have the qualities equal to the "correct" value within certain limits. As the author quotes, C. G. Darwin, Director of the National Physical Laboratory, has said, "some sort of campaign was needed to inculcate in the minds (of the engineering profession) the idea that every number has a fringe, that it is not to be regarded as exact but as so much plus or minus a bit, and that the size of this bit is one of its really important qualities."

If, therefore, we admit that a manufactured product gives rise to a frequency distribution, then statistical theory is appropriate for considering these measurements. Quality Control originated with the idea of taking certain samples off the production lines, calculating from them their statistical constants (e.g. mean and standard deviation), comparing these values with those of the samples subsequently drawn in a similar fashion, and deciding whether or not the samples vary significantly. The significant variation is itself determined by knowing the standard deviation of the mean and the standard deviation as first calculated: in accordance with the usual statistical procedure, a common practice is to take discrepancies of more than three times the standard deviation as significant, equivalent to a probability of about one in 500 (twice the standard deviation gives a probability of one in twenty, or, as it is often put, at the 5% points). At first Q.C. often took samples of order of ten or twenty and calculated means and standard deviations. Now it is usual to take samples of about five only. More would give greater sensitivity, less would give greater ease in computing. From such a sample we determine the mean and the range, and from the latter by suitable tables we have an estimate of the standard deviation. Thus, from Table C of the Appendix, we have that, for samples of five, for which the mean is \bar{x} and the range is \bar{R} , the control limits for the mean are at $\bar{x} \pm .58\bar{R}$, and for the range are 0 and $2.11\bar{R}$. If we plot as they occur the values of \bar{x} and \bar{R} , and the values lie within the limits, then the process is said to be in control, and there seems to be operating a stable system of chance causes: if not, we go "trouble-hunting". The control limits are so placed as to disclose the presence or absence of assignable causes leading to erratic variability. These assignable causes influence some, but not all, of the subgroups. The author therefore insists on the importance of "rational subgrouping", that is, of getting each subgroup as homogeneous as possible within itself to give the maximum opportunity for variation from one subgroup to another to show

itself. To be of maximum service, samples should be kept in their proper order (*e.g.* of production). British practice is to draw, in addition to the two "action" or outer control limits, two inner or warning limits: the author does not approve of this, but devotes some space to a consideration of the interpretation of "runs" of points all on the same side of the average though within the action limits.

The ideas involved and the working out lead at once to certain suggestions in the matter of tolerance limits, the idea of "natural" tolerances, and to extensions to certain special cases. They lead also to a consideration not only of some of the practices of the designer, but also to those of the inspection department, to the special problems of 100% inspection (a psychological impossibility), of destructive inspection, of limit ("go : no-go") inspection, and also to those of the consumer's acceptance procedure in relation to specification.

As a matter of statistical theory the author emphasises the desirability of measurement of qualities rather than the determination of attributes. For cases where the latter arises, by reason of demands (such as cost) other than those of statistical theory, procedures have been devised for control charts for these cases. The author gives considerable space to three of these, but points out repeatedly that they are not so sensitive as the \bar{x} and R charts: a small sample of variables is as good as a large one of attributes. The three dealt with are:

- (i) the p chart—the fraction defective;
- (ii) the c chart—the number of defects per unit;
- (iii) the $AOQL$ (average outgoing quality limit) and other sampling procedures.

This last procedure has some interesting features in what is called sequential sampling. The Operating Characteristic Curves suggest analogies with the fisherman's percentage release curve, and should be applicable to the educationalist's problem of repeated examinations for borderline candidates.

A certain amount of the theory of the various charts is given. In particular, there is rather a full explanation of the Poisson distribution in connection with the p charts, with an *abac* (p. 240, Fig. 43), and a table in the appendix (Table G). There are interesting references to certain practical devices, some due to teachers in this country, for teaching features in connection with frequency distributions and with sampling (pp. 480–482). There is a reference to Col. L. E. Simon's I_Q charts, based on Incomplete Beta Functions. Educationalists may be interested in the possibility of using some of the control chart devices for their own special purposes: a suggestion has been made above, while the \bar{x} , R charts have already been found of value in revealing differences between the members of a panel of examiners marking various scripts of the same paper. The idea of consulting mathematical statisticians for handling problems by the techniques of this book may also appeal to readers.

The book, says the author, is a working manual. It is a pity that figures and tables are identified solely by their serial reference number: there is no index to them and it is not easy to turn the pages quickly to find any particular one required. We have already referred to an important *abac* lost on p. 240: in the same way some important standard tables needed for the computing of some control limits are Tables 23 and 24, and it is impossible to locate these on pp. 210, 211 without hunting through the book. Some of the tables used in the examples have no reference number, and there are no answers to the examples. Most of the more important tables of reference

are in the appendix; among these are the tables to which reference has already been made, together with a table of areas under the normal curve, and one of logarithms of factorials (to four decimal places) for 0(1)1009. The appendix also contains a forty-item bibliography. Taken in all, however, the book is likely to be a most serviceable volume about the techniques so far developed for the practical use of control charts, especially in the field of industrial production.

FRANK SANDON.

A First Course in Mathematical Statistics. By C. E. WEATHERBURN. Pp. xv, 271. 15s. 1946. (Cambridge University Press)

Dr. Weatherburn, Professor of Mathematics at Perth, Western Australia, will be known to many readers as the author of *Differential Geometry of Three Dimensions*. The present book is based on a course on "the mathematical foundations of the interpretation of statistical data", taking the form of some sixty lectures at the University of Western Australia to graduate and undergraduate students in agriculture, biology, economics, psychology, physics and chemistry. It is a sound and workmanlike volume.

It begins with the usual topics—distributions, probability, binomial and Poisson distributions, regression and correlation (including covariance and curvilinear regression) and sampling. Moment generating functions, characteristic functions, and K -statistics (cumulative functions) are here given. Chapter VIII on Beta and Gamma functions is a feature special to the author's treatment of the subject; he uses this chapter very successfully in his later chapters on various tests and analyses. In this way he gets a very neat text with some elegant work in it. He deals rather more fully perhaps than is usual with the analysis of covariance and with non-linear-bivariate regression. There is a select bibliography and an index.

The book is dedicated to R. A. Fisher and to the Memory of Karl Pearson, and the former author especially is extensively quoted, both for ideas and for tables of functions, though there is, as suggested reading, a larger proportion than is common in Britain of text of U.S.A. treatises. In addition, the Snedecor F Test (the ratio of two variances, so-called in honour of Fisher) is used in place of our more usual Fisher z test (half the difference of the natural logarithms of the two variances); the F test does not appear to be widely known in this country, though it is perhaps easier to handle. One of the author's collaborators is Mr. D. T. Sawkins, whose papers in the *Journal and Proceedings of the Royal Society of New South Wales* on certain matters of statistical theory are probably not known to English readers.

The book is well printed and bound. Each opening is headed with the page numbers, the chapter number and section references, the chapter heading and the topic reference, so that reference is comparatively easy. Examples for the student are given at the end of each chapter. No solutions are regularly given, but the questions are phrased in nearly every case to give the answer in the text: in some there is a very full solution. The book will, however, be perhaps rather more advanced than the average undergraduate student in the sciences mentioned will be able to deal with comfortably. Other minor criticisms are (i) the assumptions about normality and size of population of sample are not always clearly stated; (ii) it is a pity that the author does not give graphs of his various functions on, e.g., pp. 149, 153 and 188; (iii) where he does give diagrams or graphs they are often (e.g. on pp. 52, 70, 88, 172 and 197) rather too sketchy with insufficient legends to be of much help to the average student. It should be noted that the terms "confidence limits" and "fiducial limits" are used synonymously.

FRANK SANDON.

The Advanced Theory of Statistics. II. By M. G. KENDALL. Pp. vii, 521. 50s. 1946. (Griffin)

Volume I of this book was reviewed in the *Mathematical Gazette*, XXVIII, No. 282 (December, 1944), pp. 223-4. The present volume is the promised completion of the work. The pagination begins afresh, but the chapter headings run on, so that we start here with Chapter 17. The system of reference numbers for sections of a chapter, tables, graphs, equations, examples (in the text) and exercises (at the chapter ends) indicates immediately in each case the chapter in which any particular item is to be found, but we find, for all that, that the necessary consultations are very awkward in the absence of any page reference.

In the Preface to Vol. I the author wrote: "In the first volume it has been possible to avoid a detailed examination of controversial topics connected with the logic of inference in probability: the subject will be taken up more systematically in the second volume." This volume accordingly begins with four chapters on estimation for the parent population from a sample. It thus deals with confidential and fiducial intervals. The author draws a distinction, not always observed in the past in the literature, between these last two, the Neyman-Pearson methods dealing with the first and the Fisher methods with the second.

Kendall distinguishes between estimation and tests of significance, the latter being the subject of his Chapter 21. "In estimation we try to find, with greater or less accuracy, the value of some parameter in a population which is known to be (or assumed to be) dependent on that parameter. In tests of significance we are given some value of a parameter beforehand and wish to decide whether it is acceptable in the light of the evidence." Chapter 22—regression—includes the theory of curvilinear regression and the use of orthogonal polynomials. This is followed by two chapters on the analysis of variance and of covariance, which, originally devised by R. A. Fisher in connection with agricultural experiments, is now of use in many different directions, especially in biological research (including medicine and psychology) and in technical manufacturing work (see, e.g., the 1946 H.M.S.O. publication, *Industrial Experimentation*, 2s. net). This leads to Chapter 25 on the Design of Sampling Inquiries, again based largely on the work of Fisher.

The following two chapters consider the idea of bias, the null hypothesis, and the "power" of a "critical region" for dealing with the probabilities of errors. Multivariate analysis is dealt with in Chapter 28, this including a development of the useful Discriminatory Analysis. Then we have two chapters in time series. These arise in many cases of observations, and, by analogy with Fourier analysis and with astronomical and tidal phenomena, search is made for periodicities of various kinds. Kendall distinguishes between oscillations, ripples, and cycles, and much of the work, analysis and illustrations are due to the writer himself. Meteorologists and economists, especially in America, are much given to these kinds of attempts at analysis and there is an extensive literature, though there is little doubt that many of the conclusions that have often been drawn, and certainly the forecasts that have been attempted by extrapolation in such cases, are unjustified.

The Appendix includes a valuable sixty-page bibliography. It is not complete: we notice—in particular—that Kendall's own revision of Yule's *Introduction to Statistics* is not listed. The Index at the end is for this volume only: Vol. I had its own index.

The volume now before us is a masterly survey of the field. Much of the work was formerly to be found only in the scattered literature and had not been coordinated before this. Some matters that we would like to have seen

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are not here : teachers in particular will regret that, after getting near the topic in his references to Hotelling and after listing appropriate works in the appendix, the topic of the psychologist's " factor analysis " does not appear. Although there is a certain amount of arithmetical treatment of experimental facts and observations, particularly in connection with the last two chapters, yet the volume will probably not be so immediately serviceable to the computing statistician as is Vol. I. Thus the Quantity Control of the industrialist, though demanding logically the survey at the beginning of this volume, will appear to the engineer to get little from this book. But the author has put all serious students of this rapidly widening subject very deeply in his debt.

The book is well produced, and misprints are extremely rare.

FRANK SANDON.

Building Geometry. By W. J. STONE. Pp. 213. 7s. 6d. 1946. Building Craft Series. (Longmans)

This book is dedicated to all students of building in the hope that, in this mechanical age, it may help them to retain their appreciation of the beauty of craftsmanship. It covers the National Building Certificate in S1, S2 and part of S3. The first eight chapters cover in outline Practical Plane and Solid Geometry with its applications in Chapter IX to Arches, Curves and Mouldings ; Chapter X to Developments, Coverings and Interpenetration of Solids ; Chapter XI to Raking Mouldings ; Chapter XII to Bevelled Work : Hopper and Roof Bevels ; Chapter XIII to the Geometry of Stairs and Hand-railing ; while Chapter XIV is called Problems and Exercises. This last chapter consists of various problems with solutions and will be welcomed by the average student. Otherwise the book is lacking in graded exercises which the student can work for himself. An increasing number of technical books, involving mathematical work, are being produced without providing exercises for the reader. Building mathematics has not received the same collaboration as in the case of engineering mathematics. Consequently much of the first eight chapters is already contained in mathematical textbooks. For instance, in Fig. 7 what we know as a rectangle is labelled as a parallelogram, and a trapezium in Fig. 12 is taken as a quadrilateral, while what we know as a trapezium is labelled as a trapezoid. A pentagon has five equal sides. Generators are called generatrices. Nevertheless, any teacher of geometry will find the drawings extremely interesting, and there is here much scope for correlation between geometry and practical drawing. The problems of (1) finding the centre of a circle through three given points and (2) of drawing a circle to touch a given line at a given point and also touch a given circle are both done by laborious approximate loci methods, whereas the ordinary geometrical methods are quite simple. The drawings of conics are also interesting, but it is the custom of this treatment to assume properties of the conics everywhere without proof. The strictly Building geometry is, however, well up to the standard which one would expect from a man of Mr. Stone's experience. There is a good chapter on instruments and accuracy in draughtsmanship. An inch is shown clearly divided into 48 equal parts, and the accuracy aimed at in drawing is to $\frac{1}{1000}$ th part of an inch.

The author is emphatic on the point that models are indispensable for the easiest and most direct approach to the subject. Chapter III is devoted to cardboard models of the cube, tetrahedron, octahedron and roofs with various applications. The object is to train the student towards acquiring the faculty of visualising solids, planes and lines in the mind's eye. It is the old difficulty, of course, of visualising three-dimensional objects by two-dimensional figures, and, later on in the book, it is evident that the author assumes this faculty. Progress would be extremely slow if a model had to be made for every problem.

Some teachers withdraw models as soon as possible so that the student can be trained to read from drawings rather than models.

The book is attractively produced, contains 213 pages and 430 well-drawn figures. With such a large number of drawings, the author has evidently been obliged to concentrate on good ones, so that the wealth of description necessary can be cut down to a minimum in order to save space. On the whole he has done the job well, but the private student may find it hard going to understand all the figures from the sometimes somewhat perforce sketchy descriptions, for example in Figs. 129-131 and Fig. 324.

Chapters I, II, IV, V and VIII could be taken together, followed by Chapters III, VI and VII. Chapter X starts off with a difficult example. The photographs on page 124 are rather small, but those on pages 174A to 174D are particularly good.

The ellipse, it is stated, is most frequently used by craftsmen in the building trades. The parabola comes next in importance, and there is one application of the hyperbola in Chapter XIV.

The author warns the reader that his treatment will differ considerably from that to be expected in a book on deductive geometry, and he has concentrated on good draughtmanship which, he maintains, can be an object of supreme beauty to anyone educated to appreciate its value. There is a wealth of experience in this low-priced book.

Altogether this is a fascinating book, and one to be recommended to all teachers of geometry and building geometry.

A. B.

CORRESPONDENCE.

PROJECTIVE GEOMETRY.

To the Editor of the *Mathematical Gazette*.

SIR,—I am grateful to Dr. E. A. Maxwell for having drawn attention, in his letter which you published on p. 303 of the *Math. Gazette*, to a point with which I might have dealt more thoroughly in my article on pp. 122, *et seq.* of Vol. XXX. I do not think, though, that any logical slip has been made.

The argument concerns a conic which is defined by five points, viz. three points of the base of a pencil of conics and the two double points of the involution created by the pencil on a straight line l . Now this conic degenerates if l connects two (appropriately chosen) diagonal points of the complete quadrangle of the base. In that case the four points of my theorem reduce to the well-known harmonic range on the diagonal of a complete quadrangle. I do not know of any reason why this special case of a general theorem should not be considered as legitimately implied, although simpler proofs may clearly be available. The present case merely requires that an equality of cross-ratios should be deemed to hold when both are zero. It seems a moot point whether such a situation can be called a breakdown, but it is in any case a breakdown of a peculiar sort, in that it leads always to correct results.

I am, Sir, Yours faithfully, S. VAJDA.

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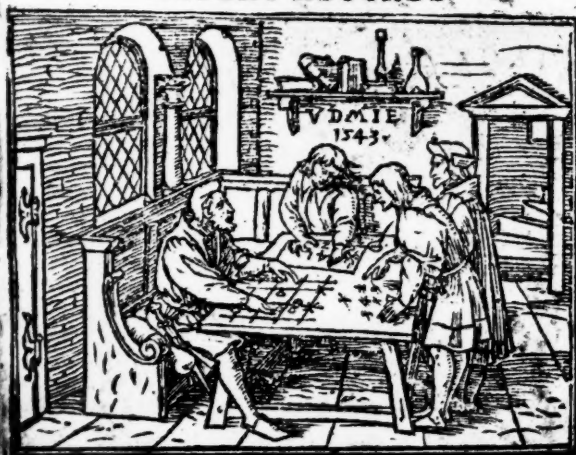
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teachyng the worke and practise of Arithmetike, moche necessary for all states of men. After a more easlyer & exacter sorte, then any lyke hath hytherto ben set forth: with dyuers newe additions, as by the table doth partly appeare.

ROBERT RECORDE.



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Vol. 2

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